Finite Embeddability Property of Distributive Lattice-ordered Residuated Groupoids with Modal Operators

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Preliminaries

Associative Lambek Calculus $L$: (Lambek 1958) ($\Gamma \neq \varepsilon$)

\[(Id)\quad A \Rightarrow A\]

\[(\backslash L)\quad \frac{\Gamma, B, \Delta \Rightarrow C \quad \Phi \Rightarrow A}{\Gamma, \Phi, A\backslash B, \Delta \Rightarrow C}\]
\[(/ L)\quad \frac{\Gamma, B, \Delta \Rightarrow C \quad \Phi \Rightarrow A}{\Gamma, B/A, \Phi, \Delta \Rightarrow C}\]
\[(\cdot L)\quad \frac{\Gamma, A, B\Delta \Rightarrow C}{\Gamma, A \cdot B, \Delta \Rightarrow C}\]

\[(CUT)\quad \frac{\Gamma, A, \Delta \Rightarrow B \quad \Phi \Rightarrow A}{\Gamma, \Phi, \Delta \Rightarrow B}\]

Nonassociative Lambek Calculus $NL$: (Lambek 1961)

Formula structures (trees): formulas, $\Gamma \circ \Delta$; Sequent: $\Gamma \Rightarrow A$

\[(\backslash L)\quad \frac{\Delta \Rightarrow A \quad \Gamma[B] \Rightarrow C}{\Gamma[\Delta \circ A\backslash B] \Rightarrow C}\]
\[(/ L)\quad \frac{\Gamma[\Delta] \Rightarrow C \quad \Delta \Rightarrow B}{\Gamma[A/B \circ \Delta] \Rightarrow C}\]
\[(\cdot L)\quad \frac{\Gamma[A \circ B] \Rightarrow C}{\Gamma[A \cdot B] \Rightarrow C}\]

\[(CUT)\quad \frac{\Delta \Rightarrow A \quad \Gamma[A] \Rightarrow B}{\Gamma[\Delta] \Rightarrow B}\]

(CUT) is admissible in $L$ and $NL$. 
A residuated semigroup: $\mathcal{M} = (M, \leq, \cdot, \backslash, /)$ s.t. $(M, \leq)$ is a poset such that $(M, \cdot)$ is semigroup $\backslash, /$ are binary operations on $M$, respectively, satisfying the residuated law:

$$\tag{1} (RES) \quad a \cdot b \leq c \iff b \leq a \backslash c \iff a \leq c / b$$

A residuated groupoid: need not be associative

A valuation $\mu$ in $\mathcal{M}$ is a homomorphism from the formula into algebra $\mathcal{M}$. A sequent $\Gamma \Rightarrow A$ is true in the model $(\mathcal{M}, \mu)$, if $\mu(\Gamma) \leq \mu(A)$.

$L$ is strongly complete w.r.t. residuated semigroups. $NL$ is strongly complete w.r.t. residuated groupoids.
CUT is not admissible in system with (D).

Distributive Full Nonassociative Lambek Calculus (DFNL) is strongly complete w.r.t. distributive lattice-ordered residuated groupoid.

A distributive lattice-ordered residuated groupoid: $(G, \wedge, \vee, \cdot, \backslash, /)$ such that $(G, \wedge, \vee)$ is a distributive lattice and $(G, \cdot, \backslash, /)$ is a residuated groupoid, where the order is lattice order.

Full Lambek Calculus (FL) is strongly complete w.r.t. lattice-ordered residuated groupoid.

Full Nonassociative Lambek Calculus (FNL) is strongly complete w.r.t. lattice-ordered residuated groupoid.

(CUT) is not admissible in system with (D).

Distributive axiom: (D) $A \wedge (B \vee C) \vdash (A \wedge B) \vee (A \wedge C)$.

(CUT) is not admissible in system with (D).

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(CUT) is not admissible in system with (D).
Modalities (MOORTGAT 1996)

\[
\begin{align*}
\text{(◊L) } & \frac{\Gamma[⟨A⟩] \Rightarrow B}{\Gamma[◊A] \Rightarrow B} \\
\text{(◊R) } & \frac{\Gamma \Rightarrow A}{⟨\Gamma⟩ \Rightarrow ◊A} \\
\text{(□ ↓ L) } & \frac{\Gamma[A] \Rightarrow B}{\Gamma[⟨□ ↓ A⟩] \Rightarrow B} \\
\text{(□ ↓ R) } & \frac{⟨\Gamma⟩ \Rightarrow A}{\Gamma \Rightarrow □ ↓ A} \\
\text{(4) } & \frac{\Gamma[⟨∆⟩] \Rightarrow A}{\Gamma[⟨⟨∆⟩⟩] \Rightarrow A} \\
\text{(T) } & \frac{\Gamma[⟨∆⟩] \Rightarrow A}{\Gamma[∆] \Rightarrow A}
\end{align*}
\]

A distributive lattice-ordered residuated groupoid with S4-operators (S4-\text{dlrg}) is a structure \((G, \wedge, \lor, ·, \setminus, /, ◊, □ ↓)\) such that \((G, \wedge, \lor)\) is a distributive lattice and \((G, ·, \setminus, /, ◊, □ ↓)\) is a structure such that ·, \setminus, / and ◊, □ ↓ are binary and unary operations on \(G\), respectively, satisfying the above conditions (1) and standard modal S4-axioms:

\[
\begin{align*}
\text{T } & a \leq ◊a, \\
\text{4 } & ◊◊a \leq ◊a \\
\text{K } & ◊(a \land b) \leq ◊a \land ◊b
\end{align*}
\]

Remark: K is admissible in S4-\text{dlrg}. Here after we slip this axiom.

DNFL\text{S}_4 is strongly complete w.r.t S4-\text{dlrg}
A class of algebras \( \mathcal{K} \) is said to have the finite embeddability property (FEP) if for every algebra \( \mathcal{A} \) in \( \mathcal{K} \) and every finite partial subalgebra \( \mathcal{B} \) of \( \mathcal{A} \), there exists a finite algebra \( \mathcal{D} \) in \( \mathcal{K} \) such that \( \mathcal{B} \) embeds into \( \mathcal{D} \).
A class of algebras $\mathcal{K}$ is said to have the finite embeddability property (FEP) if for every algebra $A$ in $\mathcal{K}$ and every finite partial subalgebra $B$ of $A$, there exists a finite algebra $D$ in $\mathcal{K}$ such that $B$ embeds into $D$.

- FEP imply the decidability of the universal theories of relative algebra.
- FEP imply consequence relation of the corresponding logic is decidable.
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- FEP of residuated groupoids
- FEP of distributive lattice-ordered residuated groupoids
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[5]. W. Buszkowski, Interpolation and FEP for Logic of Residuated Algebras, Logic Journal of the IGPL,

- FEP of RAs (residuated algebras), distributive lattice-ordered RAs, boolean RAs, Heyting RAs and double RAs
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FEP of S4-$dlrgs$ (Our results also state for $dlrgs$ with modal operators satisfying 4 or T only).
A class $\mathcal{K}$ of algebras has Strong Finite Model Property (SFMP) if every Horn clause that fails to hold in $\mathcal{K}$ can be falsified in a finite member of $\mathcal{K}$.

Strong Finite Model Property (SFMP) of a formal system $S$: if $\vdash \phi \Rightarrow A$ does not hold in $S$, then there exist a finite model of $S (\mathcal{M}, \mu)$ such that all sequents from $\Phi$ are true, but $\Gamma \Rightarrow A$ is not in $(\mathcal{M}, \mu)$.

If a formal system $S$ is strongly complete with respect to $\mathcal{K}$, then it yields, actually, an axiomatization of the Horn theory of $\mathcal{K}$; hence SFMP for $S$ with respect to $\mathcal{K}$ yields SFMP for $\mathcal{K}$.
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Strong Finite Model Property (SFMP) of a formal system $S$: if $\vdash \phi \Rightarrow A$ does not hold in $S$, then there exist a finite model of $S$ ($(M, \mu)$) such that all sequents from $\Phi$ are true, but $\Gamma \Rightarrow A$ is not in $(M, \mu)$.

If a formal system $S$ is strongly complete with respect to $\mathcal{K}$, then it yields, actually, an axiomatization of the Horn theory of $\mathcal{K}$; hence SFMP for S with respect to $\mathcal{K}$ yields SFMP for $\mathcal{K}$.

**Theorem**

*If a class of algebras $\mathcal{K}$ is closed under (finite) products, then SFMP for $\mathcal{K}$ is equivalent to FEP for $\mathcal{K}$.*
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Strong Finite Model Property (SFMP) of a formal system $S$: if $\vdash \phi \implies A$ does not hold in $S$, then there exist a finite model of $S$ $(M, \mu)$ such that all sequents from $\Phi$ are true, but $\Gamma \implies A$ is not in $(M, \mu)$.

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**Theorem**

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SFMP for $\text{DNFL}_{S_4}$(FEP of $S_4$-$drlgs$)
Linguistic analysis of modalities and additives

L or NL enriched with modalities or additive can be used to analysis some linguistic phenomenon like feature agreement, feature description, parasitic gap and so on.

Let me show some very easy example:

\( \Box \downarrow_{\text{sing}} np \) denote singular noun phrase and \( \Box \downarrow_{\text{pl}} np \) denote plural noun phrase

1. \( \text{walks} \rightarrow \Box \downarrow_{\text{sing}} np \backslash s \)
2. \( \text{walk} \rightarrow \Box \downarrow_{\text{pl}} np \backslash s \)
3. \( \text{walked} \rightarrow np \backslash s \)
4. \( \text{John} \rightarrow \Box \downarrow_{\text{sing}} np \)
5. \( \text{the Beatles} \rightarrow \Box \downarrow_{\text{pl}} np \)
6. \( \text{the Chinese} \rightarrow \Box \downarrow_{\text{sing}} \Box \downarrow_{\text{pl}} np \)

The Chinese walk. The Chinese walks.

\[
\frac{np \Rightarrow np}{\langle \square \downarrow_{pl} np \rangle \Rightarrow np} \quad (\square \downarrow L), (T)
\]

\[
\frac{\square \downarrow sing \square \downarrow_{pl} np \Rightarrow \square \downarrow_{sing} np}{\square \downarrow sing \square \downarrow_{pl} np \circ \square \downarrow sing np \sRightarrow s} \quad (\backslash L)
\]

1. become → $vp/np \lor ap$
2. wealthy → $ap$
3. and → $(ap \lor np \setminus ap \lor np)/ap \lor np$
4. a professor → $np$

become a professor and wealthy
Interpolation property

Lemma

If $\Phi \vdash_{NL} \Gamma[\Delta] \Rightarrow A$, then there exists a formula $D$ such that $\Phi \vdash_{NL} \Delta \Rightarrow D$ and $\Phi \vdash_{NL} \Gamma[D] \Rightarrow A$, where $D$ is a subformula of some formulae appearing in $\Gamma[\Delta] \Rightarrow A$ and $\Phi$.

- $NL\diamond$ (Jäger 2004) $NL\land$ (Farulewski 2008) $DFNL$ (Buszkowski, and Farulewski 2009) $NL_{S4}$ (Plummer 2008).
- The consequence relation of $NL$ is decidable in polynomial time (Buszkowski 2005).
- Context-freeness of $NL\diamond$ (Jäger 2004), $NL_{S4}$ (Plummer 2008), $DFNL$ (Buszkowski, and Farulewski).
- FEP of Rgs, Dlrgs (Farulewski 2008, Buszkowski, and Farulewski 2009), FEP of RAs, distributive lattice-ordered RAs, boolean RAs, Heyting RAs and double RAs (Buszkowski 2010).
Question:

? interpolation property for DNFL_{S4} YES

Let $T$ denote a set of formulas

- $T$-sequent: A sequent such that all formulas occurring in it belong to $T$.
- $\Phi \vdash_S \Gamma \Rightarrow_T A$: If $\Gamma \Rightarrow A$ has a deduction from $\Phi$ (in the given calculus $S$) which consists of $T$-sequents only (called a $T$-deduction).
- $T$-equivalent: Two formulae $A$ and $B$ are said to be $T$-equivalent in calculus $S$, if and only if $\vdash_S A \Rightarrow_T B$ and $\vdash_S B \Rightarrow_T A$. 
Lemma

Let $T$ be a set of formulae closed under $\lor$, $\land$. If $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[\langle \Delta \rangle] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{S4}} \langle \Delta \rangle \Rightarrow_T D$, $\Phi \vdash_{\text{DFNL}_{S4}} \langle D \rangle \Rightarrow_T D$, and $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[D] \Rightarrow_T A$. 
Lemma

Let $T$ be a set of formulae closed under $\lor$, $\land$. If $\Phi \vdash_{\text{DFNL}_{S_4}} \Gamma[\langle \Delta \rangle] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{S_4}} \langle \Delta \rangle \Rightarrow_T D$, $\Phi \vdash_{\text{DFNL}_{S_4}} \langle D \rangle \Rightarrow_T D$, and $\Phi \vdash_{\text{DFNL}_{S_4}} \Gamma[D] \Rightarrow_T A$.

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Lemma

Let $T$ be a set of formulae closed under $\lor, \land$. If $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[\langle \Delta \rangle] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{S4}} \langle \Delta \rangle \Rightarrow_T D$, $\Phi \vdash_{\text{DFNL}_{S4}} \langle D \rangle \Rightarrow_T D$, and $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[D] \Rightarrow_T A$.

Lemma

Let $T$ be a set of formulae closed under $\lor, \land$. If $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[\Delta] \Rightarrow_T A$ then there exists a $D \in T$ such that $\Phi \vdash_{\text{DFNL}_{S4}} \Delta \Rightarrow_T D$ and $\Phi \vdash_{\text{DFNL}_{S4}} \Gamma[D] \Rightarrow_T A$.

Lemma

If $T$ is set of formulas generated from a finite set and closed under $\land, \lor$, then $T$ is finite up to the relation of $T$-equivalence in $\text{DFNL}_{S4}$. 
Let $\mathcal{M} = (M, \cdot, \diamondsuit)$ be a groupoid with a unary operation $\diamondsuit$.

$$U \diamond V = \{a \cdot b \in G : a \in U, b \in V\} \quad U \setminus V = \{z \in G : U \odot \{z\} \subseteq V\}, \quad V/U = \{z \in M; \{z\} \odot U \subseteq V\}$$

$C : P(M) \to P(M)$ (4T-closure operator on $\mathcal{M}$)

- (C1) $U \subseteq C(U)$. (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$

For any $U \subseteq M$: $U$ is $C$-closed, if $C(U) = U$. $C(M)$: the family of all closed subsets of $M$. Operation on $C(M)$ are defined as follows:

$$U \otimes V = C(U \odot V), \quad \blacklozenge U = C(\blacklozenge U), \quad U \lor_C V = C(U \lor V), \quad \setminus, \quad /, \quad \blacksquare \downarrow, \quad \land, \quad \lor_C$$ as above.

**Theorem**

$C(\mathcal{M}) = (C(M), \otimes, \setminus, /, \blacklozenge, \blacksquare \downarrow, \land, \lor_C)$ is an $S_4$-lattice order residuated groupoid.
Let $\mathcal{M} = (M, \cdot, \diamond)$ be a groupoid with a unary operation $\diamond$.

- $U \odot V = \{ a \cdot b \in G : a \in U, b \in V \}$
- $U \setminus V = \{ z \in G : U \odot \{ z \} \subseteq V \}$
- $V/U = \{ z \in M ; \{ z \} \odot U \subseteq V \}$
- $\diamond U = \{ \diamond a \in M | a \in U \}$
- $\square \downarrow U = \{ z \in M | \diamond z \in U \}$

$C : P(M) \rightarrow P(M) \ (4T\text{-closure operator on } \mathcal{M})$

- (C1) $U \subseteq C(U)$.
- (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$
- (C3) $C(C(U)) \subseteq C(U)$.
- (C4) $C(U) \odot C(V) \subseteq C(U \odot V)$

For any $U \subseteq M$: $U$ is $C$-closed, if $C(U) = U$. $C(M)$: the family of all closed subsets of $M$. Operation on $C(M)$ are defined as follows:

$U \otimes V = C(U \odot V)$, $\blacklozenge U = C(\diamond U)$, $U \lor_C V = C(U \lor V)$, $\setminus, /, \square \downarrow, \land, \lor_C$ as above.

**Theorem**

$C(\mathcal{M}) = (C(M), \otimes, \setminus, /, \blacklozenge, \square \downarrow, \land, \lor_C)$ is an $S_4$-lattice order residuated groupoid.
Let $\mathcal{M} = (M, \cdot, \Diamond)$ be a groupoid with a unary operation $\Diamond$.

- $U \odot V = \{a \cdot b \in G : a \in U, b \in V\}$
- $U \backslash V = \{z \in G : U \odot \{z\} \subseteq V\}$
- $V/U = \{z \in M : \{z\} \odot U \subseteq V\}$

- $\Diamond U = \{\Diamond a \in M | a \in U\}$
- $\Box \downarrow U = \{z \in M | \Diamond z \in U\}$

- $U \vee V = U \cup V$, $U \wedge V = U \cap V$

$C : P(M) \rightarrow P(M)$ (4T-closure operator on $\mathcal{M}$)

- (C1) $U \subseteq C(U)$.
- (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$
- (C3) $C(C(U)) \subseteq C(U)$.
- (C4) $C(U) \odot C(V) \subseteq C(U \odot V)$
- (C5) $\Diamond C(U) \subseteq C(\Diamond U)$

For any $U \subseteq M$: $U$ is $C$-closed, if $C(U) = U$. $C(M)$: the family of all closed subsets of $M$. Operation on $C(M)$ are defined as follows:

- $U \otimes V = C(U \odot V)$,
- $\Diamond U = C(\Diamond U)$,
- $U \vee_C V = C(U \vee V)$,
- $\backslash$, $/$, $\Box \downarrow$, $\land$, $\lor$ as above.

**Theorem**

$C(\mathcal{M}) = (C(M), \otimes, \backslash$, $/$, $\Diamond$, $\Box \downarrow$, $\land$, $\lor_C)$ is an $S_4$-lattice order residuated groupoid.
Let $\mathcal{M} = (M, \cdot, \diamond)$ be a groupoid with a unary operation $\diamond$.

- $U \odot V = \{a \cdot b \in G : a \in U, b \in V\}$
- $U \\setminus V = \{z \in G : U \odot \{z\} \subseteq V\}$
- $V/U = \{z \in M ; \{z\} \odot U \subseteq V\}$
- $\diamond U = \{\diamond a \in M | a \in U\}$
- $\Box \downarrow U = \{z \in M | \diamond z \in U\}$
- $U \lor V = U \cup V, U \land V = U \cap V$

$C : P(M) \to P(M)$ (4T-closure operator on $\mathcal{M}$)

(C1) $U \subseteq C(U)$. (C2) if $U \subseteq V$ then $C(U) \subseteq C(V)$
(C3) $C(C(U)) \subseteq C(U)$. (C4) $C(U) \odot C(V) \subseteq C(U \odot V)$
(C5) $\diamond C(U) \subseteq C(\diamond U)$
(C6) $C(\diamond C(\diamond C(U))) \subseteq C(\diamond U)$. (C7) $C(U) \subseteq C(\diamond U)$

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$U \otimes V = C(U \odot V)$, $\ast_U = C(\diamond U)$, $U \lor_C V = C(U \lor V)$, $\setminus$, $/ \Box \downarrow, \land$ as above.

Theorem

$C(\mathcal{M}) = (C(M), \otimes, \setminus, /, \ast, \Box \downarrow, \land, \lor_C)$ is an $S_4$-lattice order residuated groupoid.
$T$: nonempty set of formulae containing all subformulae of formulae in $\Phi$; $T^*$: all formula structures form out of formulae in $T$. Similarly; $T^* [\circ]$: all contexts in which all formulae belong to $T$.

Let $\Gamma[\circ] \in T^*$ and $A \in T$; $B(T)$: the family of all sets $[\Gamma[\circ], A]$

$$[\Gamma[\circ], A] = \{ \Delta : \Delta \in T^* \text{ and } \Phi \models_{DFNLS_4} \Gamma[\Delta] \Rightarrow_T A \}$$

$$C_T(U) = \bigcap \{ [\Gamma[\circ], A] \in B(T) : U \subseteq [\Gamma[\circ], A] \}$$

**Lemma**

$C_T$ is a S4-modal closed operator.

$T$: containing all formulae in $\Phi$, closed under subformulae, $\wedge$ and $\lor$. $G(T^*) = (T^*, \circ, \langle \rangle)$: a groupoid, $\langle \rangle$ is an unary operation on $T^*$.

**Lemma**

$C_T(G(T^*))$ is a S4-lrg
\( \mu : \mu(p) = [p]. \)

\[
\begin{align*}
\Diamond [A] &= \Diamond A & \Box \downarrow [A] &= \Box \downarrow A \\
\end{align*}
\] (4)

all formulas appearing in them belong to \( T \).

**Lemma**

*For any nontrivial closed set \( U \in C_T(G(T^*)) \), there exists a formula \( A \in R \) such that \( U = [A] \).*

**Lemma**

\( C_T(G(T^*)) \) is a finite \( 4T - dlr g \).
Lemma

$T$ denotes a set of formulae, containing all formulae in $\Phi$ and closed under $\land$, $\lor$, and subformulae. Let $\mu$ be a valuation in $C_T(G(T^*))$ such that $\mu(p) = [p]$. For any $T$-sequent $\Gamma \Rightarrow A$, this sequent is true in $(C_T(G(T^*)), \mu)$ if and only if $\Phi \vdash_{DFNL_{S4}} \Gamma \Rightarrow_T A$.

Theorem

Assume that $\Phi \vdash_{DFNL_{S4}} \Gamma \Rightarrow A$ does not hold. Then there exist a finite distributive lattice ordered residuated groupoid with $4T$-operators $G$ and a valuation $\mu$ such that all sequents from $\Phi$ are true but $\Gamma \Rightarrow A$ is not true in $(G, \mu)$.

Corollary

$S4 - dlr gs$ has FEP.
References


Thank you