The word problem for ΣΠ-categories, i.e. properties of free ΣΠ-categories (II)

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Cape Town, Category Theory 2009

Outline

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Introductory remarks

- ΣΠ-categories and their theory
- Context, some motivations
- The free ΣΠ-category

"En route" towards a decision procedure

- The CT2006 results

Softness, cardinals, and (some hints on) a decision procedure

- Understanding softness
- Pushouts-pullbacks, bouncing

ΣΠ-categories and their theory

A ΣΠ-*category* :
a category with (chosen) finite products and finite coproducts.

- *Binary products*:

$$\begin{array}{ccc} [X,A]\times[X,B] & \xrightarrow{\langle\,,\,\rangle} & [X,A\times B] \\ [X_i,A] & \xrightarrow{\pi_i} & [X_0\times X_1,A] \end{array}$$

  such that

$$\pi_i(\langle f_0,f_1\rangle)=f_i\,, \qquad \langle\pi_0(f),\pi_1(f)\rangle=f\,.$$

- *Binary coproducts*: dual nats $\{\,,\,\}$ and σ_j, $j=0,1$, ...
  ...and dual equations.

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Units, empty products and coproducts, terminal and initial objects

- The units:

$$[X, 1] = \{ !_X \} \qquad [0, A] = \{ ?_A \}$$

- Some derived equations:

$$\begin{aligned} \{!_X, !_Y\} &= !_{X+Y} && : X + Y \to 1 \\ !_0 &= ?_1 && : 0 \to 1 \end{aligned}$$

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The word problem for ΣΠ-categories?

- Elementary but important mathematical theory,
- Ties to linear logic (additives),
- A challenge for category theorists and logicians:
  - Joyal & Hu 96, Hu 98,
  - Dosen 99, Dosen & Petric 2009,
  - Hughes 2000, Hughes & van Glabbek 03,
  - Joyal 95,
  - Cockett & Seely 2001,
- From Whitman's condition to softess:
  mimicking the theory of free lattices,
- Joyal's theory of communication:
  free ΣΠ-categories are semantics for communication protocols,
  we tackle equivalence of these protocols.

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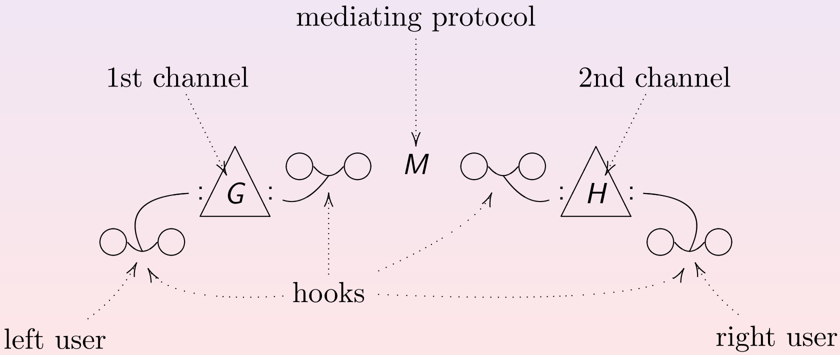
ΣΠs: a theory of communication

Objects of $\Sigma\Pi(\mathcal{A})$ are channels,

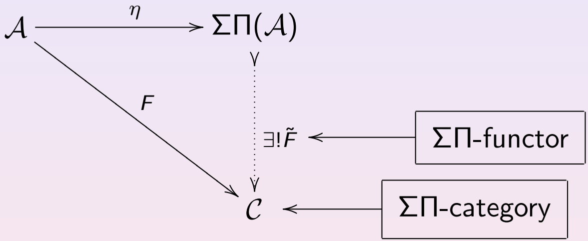
arrows of $\Sigma\Pi(\mathcal{A})$ are mediating protocols.

$$G \xrightarrow{M} H$$



$\Sigma\Pi(\mathcal{A})$, the free $\Sigma\Pi$ category on $\mathcal{A}$

- *free* $\Sigma\Pi$-category generated by $\mathcal{A}$:



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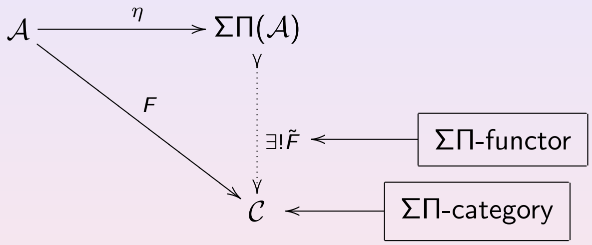

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CS01 : a Lambek-style construction of $\Sigma\Pi(\mathcal{A})$

$$\frac{x \xrightarrow{f} y}{\eta(x) \xrightarrow{\eta(f)} \eta(x)}$$

$$\frac{-}{X \xrightarrow{!} 1}\, R1$$

$$\frac{X_i \xrightarrow{f} A}{X_0 \times X_1 \xrightarrow{\pi_i(f)} A}\, L_i\times \qquad \frac{X \xrightarrow{f} A \quad X \xrightarrow{g} B}{X \xrightarrow{\langle f,g\rangle} A \times B}\, R\times$$

$$\frac{-}{0 \xrightarrow{?} A}\, L0$$

$$\frac{X \xrightarrow{f} A \quad Y \xrightarrow{g} A}{X + Y \xrightarrow{\{f,g\}} A}\, L+ \qquad \frac{X \xrightarrow{f} A_j}{X \xrightarrow{\sigma_j(f)} A_0 + A_1}\, R_j+$$

CS01: confluence modulo equations

Proposition

The cut-elimination procedure is confluent modulo the equations:

$$\pi_i(\langle f, g\rangle) = \langle \pi_i(f), \pi_i(g)\rangle \quad \sigma_j(\{f, g\}) = \{\sigma_j(f), \sigma_j(g)\}$$

$$\pi_i(\sigma_j(f)) = \sigma_j(\pi_i(f))$$

$$\{\langle f_{11}, f_{12}\rangle, \langle f_{21}, f_{22}\rangle\} = \langle\{f_{11}, f_{21}\}, \{f_{12}, f_{22}\}\rangle$$

$$\pi_i(!) = ! \qquad \sigma_j(?) = ?$$

$$\{!, !\} = ! \qquad \langle ?, ?\rangle = ?$$

$$!_0 = ?_1$$

Abstract characterization of free ΣΠ-cats [CS01, J95]

1. The functor $\eta : \mathcal{A} \to \Sigma\Pi(\mathcal{A})$ is full and faithful.
2. Generators are *atomic*:

$$\coprod_j [\,\eta(a), Y_j\,] \to [\,\eta(a), \coprod_j Y_j\,]$$

$$\coprod_i [\,X_i, \eta(b)\,] \to [\,\prod_i X_i, \eta(b)\,]$$

   are isomorphisms.

3. $\Sigma\Pi(\mathcal{A})$ is *soft*:

$$\begin{array}{ccc} \coprod_{i,j} [\,X_i, Y_j\,] & \longrightarrow & \coprod_j [\,\prod_i X_i, Y_j\,] \\ \downarrow & & \downarrow \\ \coprod_i [\,X_i, \coprod_j Y_j\,] & \longrightarrow & [\,\prod_i X_i, \coprod_j Y_j\,] \end{array}$$

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Characterization of free ΣΠ-categories (cont.)

Theorem

The pair $(\eta, \Sigma\Pi(\mathcal{A}))$ *satisfies 1,2,3.*

If $\mathcal{C}$ *is a* ΣΠ*-category "generated" by* $\mathcal{A}$, $F : \mathcal{A} \to \mathcal{C}$,
and $(F, \mathcal{C})$ *satisfies 1,2,3,*
then the extension $\hat{F} : \Sigma\Pi(\mathcal{A}) \to \mathcal{C}$ *is an equivalence.*

Outline

A result presented at CT06

Theorem

In $\Sigma\Pi(\mathcal{A})$ coproducts are weakly disjoint:

$$\begin{array}{ccc} X & \xrightarrow{g} & Y_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma_1} \\ Y_0 & \xrightarrow{\sigma_0} & Y_0 + Y_1 \end{array}$$

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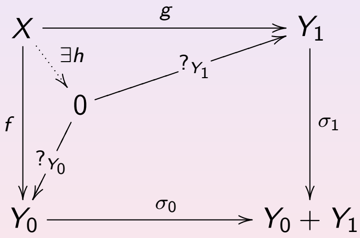

. . . and some Corollaries

- decide in linear time whether
  - an object of $\Sigma\Pi(\mathcal{A})$ is isomorphic to 0 or 1,
  - an arrow of $\Sigma\Pi(\mathcal{A})$ factors through 0 and or 1.

- a simple characterization of monic coproduct injections:

$$\sigma_0 : A \longrightarrow A + B$$

is monic iff either B is not pointed or A is pointed.

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Softness: the focus of a decision procedure

A decision procedure focuses on the homset $[X \times Y, A + B]$.

For example:

```
let equal f g : X ⟶ A × B =
  let
    f = ⟨f1, f2⟩
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Understanding softness

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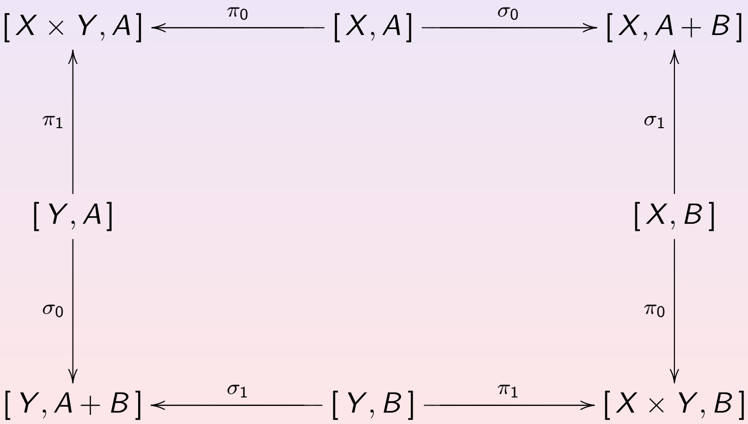
the pushout

$$\begin{array}{ccc} [\,X, A\,] + [\,X, B\,] + [\,Y, A\,] + [\,Y, B\,] & \longrightarrow & [\,X \times Y, A\,] + [\,X \times Y, B\,] \\ \downarrow & & \downarrow \\ [\,X, A + B\,] + [\,Y, A + B\,] & \longrightarrow & [\,X \times Y, A + B\,] \end{array}$$

Understanding softness

The homset $[\,X \times Y, A + B\,]$ is ...

the colimit of the "diagram of cardinals":



Understanding softness

The homset $[\, X \times Y, A + B \,]$ is ...

the quotient of

$$[\, X, A + B \,] + [\, Y, A + B \,] + [\, X \times Y, A \,] + [\, X \times Y, B \,]$$

under the equivalence relation generated by elementary pairs (f, g):

$$\begin{array}{ccc} & h \in [\, X, A \,] & \\ \swarrow^{\pi_0} & & \searrow^{\sigma_0} \\ f \in [\, X \times Y, A \,] & & g \in [\, X, A + B \,] \end{array}$$

$$f = \pi_0(h) \qquad\qquad \sigma_0(h) = g$$

(In)definite arrows

Definition

An arrow is *indefinite* if it factors through 0 or through 1.
Otherwise, it is *definite*.

A simple decision procedure for indefinite maps:

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let equal f g = (*f g factor through 0*)
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    f = f';?
  and
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  in
    if cod f is pointed then true
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      f' = g'
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A useful Lemma

If

$$\pi : f = f_0 g_1 f_2 \ldots g_n = g$$

is a path of elementary pairs crossing a corner,

then $[f] = [g] \in [\, X \times Y, A + B \,]$ is indefinite.

Consider

$$\ldots f_{i-1} = \pi_0(h_i) \quad \sigma_0(h_i) = g_i \quad g_i = \sigma_1(h_{i+1}) \quad \pi_0(h_{i+1}) \quad = f_{i+1} \ldots$$

then h_i and h_{i+1} are copointed and $[g] = [f]$ as well.

Lemma

If $[f] \in [\, X \times Y, A + B \,]$ is definite and $\pi : f \xrightarrow{*} g$ is a path of elementary pairs, then π "bounces" along one side of this diagram.

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Softness for definite maps

The previous Lemma transforms – for definite maps – the cardinal diagram from

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\begin{array}{ccccc}
[X \times Y, A] & \xleftarrow{\pi_0} & [X, A] & \xrightarrow{\sigma_0} & [X, A + B] \\
\uparrow{\scriptstyle \pi_1} & & & & \uparrow{\scriptstyle \sigma_1} \\
[Y, A] & & & & [X, B] \\
\downarrow{\scriptstyle \sigma_0} & & & & \downarrow{\scriptstyle \pi_0} \\
[Y, A + B] & \xleftarrow{\sigma_1} & [Y, B] & \xrightarrow{\pi_1} & [X \times Y, B]
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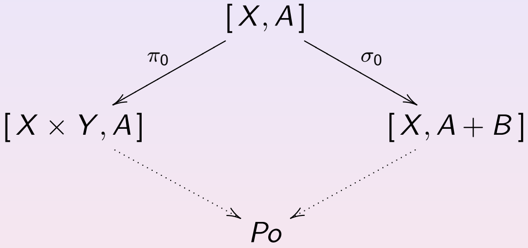
[Y, A] & & [X, B] \\
\downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \pi_0} \\
[Y, A + B] & & [X \times Y, B]
\end{array}$$

$$[Y, A + B] \xleftarrow{\sigma_1} [Y, B] \xrightarrow{\pi_1} [X \times Y, B]$$

A new main result

Theorem

A pushout diagram



is also a pullback.

That is: given $\pi : f \xrightarrow{*} g$ a path "bouncing" on the upper row, there exists a unique h such that

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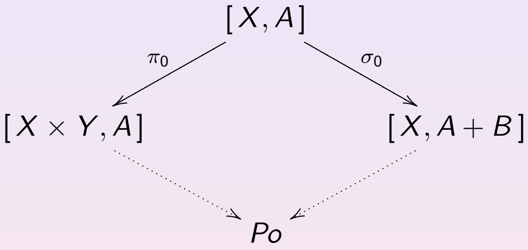

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Deciding whether $[f] = [g]$ in $[X_0 \times X_1, A_0 + A_1]$

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let equivalent f g =
  (* f g are both definite
     f : X × Y → A,  g : X → A + B  *)
  find h s.t.
    f = σ0(h) && π0(h) = g
```

Uniqueness of such h "makes it is easy" to find it.

Theorem

There exists an algorithm to decide whether two parallel arrow-terms f, g of $\Sigma\Pi(\mathcal{A})$ are equal in $[X, A]$. The procedure runs in time polynomial in

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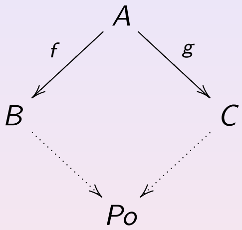
$$(\mathrm{hgt}(X) + \mathrm{hgt}(A)) \cdot \mathrm{size}(X) \cdot \mathrm{size}(A)\,.$$

Background on bouncers

Lemma

Consider a pushout in Set



TFAE:

- *the diagram is a weak pullback,*
- *$Ker(f)$ and $Ker(g)$ commute.*

A *bouncer* for $(a_0, a_2) \in A$ is $a_1 \in A$ such that

$$(a_0, a_1) \in Ker(f), \qquad (a_1, a_2) \in Ker(g).$$

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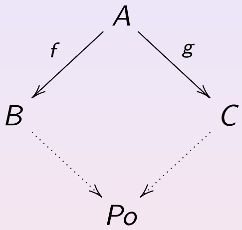


TFAE:

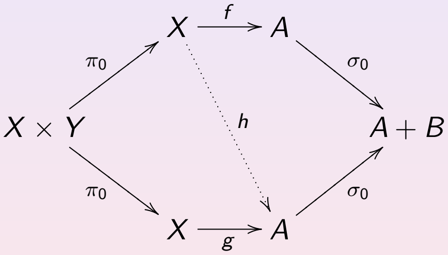
- *the diagram is a weak pullback,*
- *$Ker(f)$ and $Ker(g)$ commute.*

A *bouncer* for $(a_0, a_2) \in A$ is $a_1 \in A$ such that

$$(a_0, a_1) \in Ker(f)\,, \qquad (a_1, a_2) \in Ker(g)\,.$$

Bouncers within $\Sigma\Pi(\mathcal{A})$

A *bouncer* for $(f, g) \in [X, A]$ is an $h \in [X, A]$ such that



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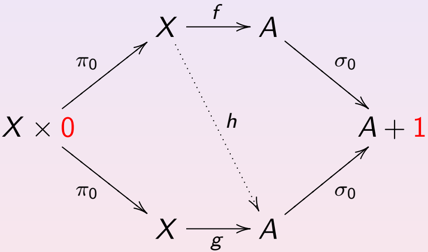

A minimal nontrivial bouncer

In $\Sigma\Pi(\emptyset)$ all bouncer are trivial. This is not the case in general:

Proposition

In $\Sigma\Pi(\mathcal{A})$ (f, g) have a bouncer if and only if (g, f) have a bouncer.

Proposition

In $\Sigma\Pi(\mathcal{A})$ there exists at most one bouncer for (f, g).

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$$\begin{matrix} f = \begin{pmatrix} z & \pi_0(?) \\ \sigma_0(!) & id_x \end{pmatrix} \\ \\ g = \begin{pmatrix} z & \pi_1(?) \\ \sigma_1(!) & id_x \end{pmatrix} \end{matrix} : (0 \times 0) + x \longrightarrow (1 + 1) \times x$$

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