# Tools for Completeness for Flat Modal Fixpoint Logics

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# Outline

## Context: logics of computation

## Axiomatizing Flat Fixpoint Logics

- Flat fixpoint logics – as extensions of $\mathbb{K}$
- ... – as fragments of $\mathcal{L}_\mu$

## Algebraic logic perspective

- Propositional modal logic as an algebraic system
- The nonsense path to completeness

## Three key properties

- Residuation
- Constructiveness
- Regularity

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# Logic in verification

Let

- $\Pi$ be a program,
- $s$ be a state of the program,
- $\phi$ be a speficication.

Does

$\Pi$ satisfy the specification $\phi$ if started from $s$ ?

# Logic in verification

Let

- $\Pi$ be a transition system,
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Does

$$\Pi \text{ satisfy the specification } \phi \text{ if started from } s\ ?$$

# Logic in verification

Let

- $\Pi$ be a transition system,
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- $\phi$ be a formula.

Does

$$\Pi, s \models \phi \text{ ?}$$

# From multimodal logic . . .

The formulas of multimodal logic $\mathbb{K}$

$$\begin{aligned}\phi = \; & x \mid \neg\phi \mid \phi \wedge \phi \\ & \mid \langle a \rangle \phi \,,\end{aligned}$$

$a \in Act$, talk about transition systems

$$\Pi = \langle\, S \,,\, \{\, \xrightarrow{a} \subseteq S \times S \mid a \in Act \,\} \,\rangle \,,$$

where

$$\Pi, s \models \langle a \rangle \phi \quad \text{iff} \quad \exists s' s \xrightarrow{a} s' \text{ and } \Pi, s' \models \phi.$$

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# ...to the logics of programs

Example:

$$\Pi, s \models \neg(\bigvee_{a \in Act} \langle a \rangle \top) \quad \text{iff} \quad s \text{ is a deadlock},$$

Limit:
there is no nultimodal formula $\phi$ such that

$$\Pi, s \models \phi \quad \text{iff} \quad \Pi \text{ can reach a deadlock from } s.$$

However:

$$\Pi, s \models LFP_x.(\neg(\bigvee_{a \in Act} \langle a \rangle \top) \vee \bigvee_{a \in Act} \langle a \rangle x)$$
$$\text{iff} \quad \Pi \text{ can reach a deadlock from } s.$$

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# A slogan on logic of programs

A logic of programs = multimodal logic $\mathbb{K}$ + some extremal fixpoints .

An example, $\mathbb{CTL}$:

$$\mathcal{L}(\mathbb{CTL}) = \mathcal{L}(\mathbb{K}) + \{ \mathcal{U} \}$$

Meaning of $\mathcal{U}$:

$$\begin{aligned} p\mathcal{U}q &\equiv \text{“}p \text{ holds until } q\text{”} \\ &\equiv \mathit{LFP} \text{ of } \gamma(x, p, q) = q \vee (p \wedge [\,]x)\,. \end{aligned}$$

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# (Very rough) State of art

Computational aspects:

- satisfaction, satisfiability,

and expressivity issues:

- hierarchies of logics, hierarchies within a logic,

are well studied.

Theoretical aspects:

- proof systems, cut-elimination,
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# An example: $\mathbb{CTL}$, a flat fixpoint logic

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# Flat fixpoint logics

Let

$$\Gamma = \{\, \gamma_i(x) \mid i = 1, \ldots, n \,\},$$

where $\gamma_i(x) \in \mathcal{L}(\mathbb{K})$, and consider

$$\mathcal{L}(\Gamma) = \mathcal{L}(\mathbb{K}) + \{\, \sharp_\gamma \mid \gamma \in \Gamma \,\}.$$

Intended meaning:

$$\sharp_\gamma \equiv \textit{LFP} \text{ of } \gamma(x).$$

**Problem**

*Find a complete axiomatization of $\mathcal{L}(\Gamma)$ w.r.t.*
*the standard semantics of Kripke frames (transition systems).*

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*Find a complete uniform axiomatization of $\mathcal{L}(\Gamma)$ w.r.t. the standard semantics of Kripke frames (transition systems).*

# The usual axiomatization (Park and Kozen)

Axioms for modal logic $\mathbb{K}$, plus

$$\vdash \gamma(\sharp_\gamma) \to \sharp_\gamma\,, \qquad (\sharp_\gamma\text{-prefix})$$

$$\text{from } \vdash \gamma(\phi) \to \phi \text{ infer } \vdash \sharp_\gamma \to \phi\,. \qquad (\sharp_\gamma\text{-least})$$

- For many flat fixpoint logics yes, e.g. $\mathbb{CTL}$.
- For many others, we don't know, but ...

... we propose another axiomatization ...

... and prove it is sound and complete.

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# The problem upside down: flat fixpoint logics – as fragments of the modal $\mu$-calculus

The language of $\mathcal{L}_\mu$:

$$\begin{aligned}\phi = x \mid \neg x \mid \top \mid \phi \wedge \phi \mid \bot \mid \phi \vee \phi \\ \mid \langle\ \rangle\phi \mid [\ ]\phi \\ \mid \mu x.\phi \mid \nu x.\phi\,. \end{aligned} \qquad (x \text{ positive in } \phi.)$$

Remark:

$$\mathbb{CTL} \subseteq \mathcal{L}_\mu\,, \qquad \mathcal{L}(\Gamma) \subseteq \mathcal{L}_\mu\,.$$

**Theorem (Kozen, Walukiewicz)**

*The usual axiomatizarion of $\mathcal{L}_\mu$ is complete w.r.t. Kripke frames.*

**Problem**

*Fragments of $\mathcal{L}_\mu$ for which the usual axiomatization is complete?*
*Alternative axiomatizations of fragments ensuring completeness?*

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# Modal $\sharp$-algebras

## Definition

- *Modal algebra* (i.e. algebraic model of modal logic $\mathbb{K}$):

$$\mathfrak{A} = \langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \langle\ \rangle^{\mathfrak{A}} \rangle$$

where $\langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}} \rangle$ is a Boolean algebra, and $\langle\ \rangle^{\mathfrak{A}}$ is normal:

$$\langle\ \rangle^{\mathfrak{A}}(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{A}}, \ \ \langle\ \rangle^{\mathfrak{A}}(x \vee^{\mathfrak{A}} y) = \langle\ \rangle^{\mathfrak{A}}(x) \vee^{\mathfrak{A}} \langle\ \rangle^{\mathfrak{A}}(y)\,.$$

- *Modal $\sharp$-algebras*:

$$\mathfrak{A} = \langle A, \bot^{\mathfrak{A}}, \vee^{\mathfrak{A}}, \neg^{\mathfrak{A}}, \langle\ \rangle^{\mathfrak{A}}, \{\, \sharp_\gamma^{\mathfrak{A}} \mid \gamma \in \Gamma \,\} \rangle$$

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# Kripke modal ♯-algebras

If $\mathcal{M} = \langle S, \to \rangle$ is a Kripke frame, then

$$\mathfrak{M} = \langle \mathcal{P}(S), \emptyset, \cup, \overline{(\,\cdot\,)}, \langle\ \rangle \rangle$$

where

$$\langle\ \rangle(X) = \{\, s \in S \mid \exists s' \in X \text{ s.t. } s \to s' \,\}$$

is a modal algebra.

It is also a modal ♯-algebra:

$\mathcal{P}(S)$ is a complete lattice hence, by Tarski-Knaster,

the LpFP of $\gamma^{\mathfrak{M}}$ exists and is uniquely determined.

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# Stone type completeness theorems

Proposition

*Every Boolean algebra embeds into the powerset algebra of its ultrafilters.*

Proposition

*Every modal algebra embeds into a the powerset algebra of some Kripke frame.*

However:

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*There is a modal $\sharp$-algebra that cannot be embedded into any complete modal $\sharp$-algebra.*

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# The nonsense path to completeness

## Proposition

*Each Kripke $\sharp$-algebra is residuated, constructive, and regular.*

## Proposition

*The free regular modal $\sharp$-algebra is residuated and constructive.*

## Proposition

*Each countable, residuated, constructive $\sharp$-algebra can be embedded into a Kripke $\sharp$-algebra.*

## Corollary (Completeness)

*If a formula is valid in every Kripke frame, then it is also derivable in a formal systems in which regularity is admissible.*

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- Constructiveness
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# Residuation or adjointness

Definition

$f : L \longrightarrow M$ is a *left adjoint* or residuated iff there exists $g : M \longrightarrow L$ such that

$$f(x) \leq y \text{ iff } x \leq g(y)\,.$$

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### Example

Let $M = \langle S, \rightarrow \rangle$ and consider

$$g(Y) = \{ s \in S \mid \forall s' \rightarrow s \text{ implies } s' \in Y \} .$$

Then $g$ is a witness that $\langle \ \rangle^{\mathfrak{M}}$ is residuated.

### Definition

A modal $\sharp$-algebra $\mathfrak{A}$ is *residuated* if $\langle \ \rangle^{\mathfrak{A}}$ is residuated.

# Residuation or adjointness, $\mathcal{O}_{\mathfrak{f}}$-adjointness

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$f : L \longrightarrow M$ is a *left adjoint* or residuated iff there exists $g : M \longrightarrow L$ such that

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**Definition**

$f : L \longrightarrow M$ is a (left) $\mathcal{O}_{\mathfrak{f}}$*-adjoint* iff there exists $G : M \longrightarrow \mathcal{P}_{\mathfrak{f}}(L)$ such that

$$f(x) \leq y \text{ iff } \exists z \in G(y) \text{ s.t. } x \leq z\,.$$

# Constructiveness

## Definition

We say that the LpFP $\mu.f$ of $f : L \longrightarrow L$ is *constructive* if

$$\mu.f = \bigvee_{\alpha \in Ord} f^{\alpha}(\bot)\,.$$

## Definition

A modal $\sharp$-algebra $\mathfrak{A}$ is *constructive* if

$$\sharp_{\gamma}^{\mathfrak{A}} = \bigvee_{\alpha \in Ord} (\gamma^{\mathfrak{A}})^{\alpha}(\bot)\,, \qquad \gamma \in \Gamma\,.$$

## Example

A Kripke $\sharp$-algebra is contructive.

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# $\mathcal{O}_{\mathfrak{f}}$-adjoints and constructiveness

**Proposition**

*If $f : L \longrightarrow L$ is a finitary $\mathcal{O}_{\mathfrak{f}}$-adjoint, then the LpFP of $f$, if it exists, is $\omega$-constructive:*

$$\mu.f = \bigvee_{n \geq 0} f^n(\bot)\,.$$

**Definition**

An $\mathcal{O}_{\mathfrak{f}}$-adjoint $f : L \longrightarrow L$ is finitary if the set

$$\bigcup_{n \geq 0} G^n(x)$$

is finite, for each $x \in L$.

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*Let $\mathfrak{A}$ be a modal algebra and suppose that*

- $f : \mathfrak{A} \longrightarrow \mathfrak{A}$ *is a finitary left adjoint,*
- *the least fixpoint*

$$f^*(y) = \mu_x.(y \vee f(x))$$

*exists in $\mathfrak{A}$.*

*Then the relation*

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*holds.*

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*Let $\mathfrak{A}$ be a modal algebra and suppose that*

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# Proof of the Proposition

If $f^n(y) \leq z$, $n \geq 0$, then $y \leq g^n(z)$, $n \geq 0$, and

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# Finitary $\mathcal{O}_{\mathfrak{f}}$-adjoints in the free modal $\sharp$-algebra.

Let

$$\nabla X = \bigwedge_{x \in X} \langle\ \rangle x \wedge [\ ] \bigvee X \,.$$

**Proposition**

*Let $\mathfrak{F}$ be the free modal $\sharp$-algebra (i.e. the Lindenbaum algebra); let $\Lambda$ be a finite set of literals. The following condition holds in $\mathfrak{F}$:*

$$\bigwedge \Lambda \wedge \nabla^{\mathfrak{F}} X \leq \bot^{\mathfrak{F}}$$

*implies*

$$\exists l \in \Lambda \; s.t. \; \neg l \in \Lambda \quad or \quad \exists x \in X \; s.t. \; x \leq \bot^{\mathfrak{F}} \,.$$

**Corollary**

*Operations in the clone of $\langle\ \rangle$, $[\ ]$, $\nabla$, $\vee$, are finitary $\mathcal{O}_{\mathfrak{f}}$-adjoints of the free modal $\sharp$-algebra. Hence their LpFPs are constructive.*

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# Remark:
# a cut-elimination theorem and the subformula property

Rephrasing of the previous Proposition:

## Proposition

*Let* $\Lambda$ *be a finite set of literals and let* $\psi, \phi_i$ *modal formulas,* $i = 1, \dots, n$*. The following sound deductive rules:*

$$\frac{\Lambda \vdash}{\Lambda, \langle\ \rangle\phi_1, \dots, \langle\ \rangle\phi_n, [\ ]\psi \vdash}\, W_\Lambda$$

$$\frac{\phi_i, \psi \vdash}{\Lambda, \langle\ \rangle\phi_1, \dots, \langle\ \rangle\phi_n, [\ ]\psi \vdash}\, L_i\langle\ \rangle + W$$

*are jointly reversible.*

# Dealing with conjunctions: the subset construction

**Lemma**

*Given* $\{\, \gamma_x \in \mathcal{L}(\mathbb{K}) \,\}_{x \in X}$, *there exists* $\{\, \delta_Y \in \mathcal{L}(\mathbb{K}) \,\}_{Y \subseteq X, Y \neq \emptyset}$ *such that*

- $\delta_y$ *is in disjunctive normal form, i.e. constructed out of* $\vee$ *and special conjuctions:*

$$\bigwedge \Lambda \wedge \nabla X\,, \qquad\qquad \Lambda \text{ a finite set of literals}\,,$$

- *the relation*

$$\bigwedge_{x \in Y} \gamma_x = \delta_Y[\bigwedge Z/Z]$$

*holds in every modal algebra.*

# Algebraic interpretation of the Lemma

Let

$$\mathcal{P}_+(X) = \{\, Y \subseteq X \mid Y \neq \emptyset \,\}, \qquad \iota_Y(\vec{v}) = \bigwedge_{x \in Y} \vec{v}_x \,.$$

On every modal algebra $\mathfrak{A}$:

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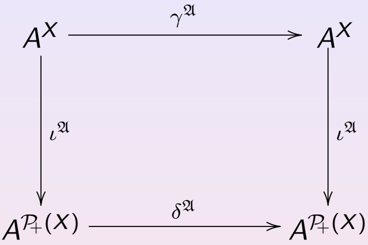
# Dealing with conjunctions: the Transfer Lemma




**Lemma (see Arnold & Niwinski)**

*If $\mathfrak{A}$ is a Kripke $\sharp$-algebra, then*

$$\mu.\delta^{\mathfrak{A}} = \iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}})\,. \tag{REG}$$

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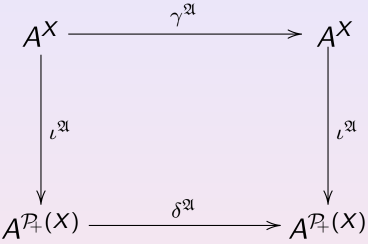


## Proposition

*If the least fixed point $\mu.\delta^{\mathfrak{A}}$ exists and is constructive, then the least fixed point $\mu.\gamma^{\mathfrak{A}}$ is also constructive, and*

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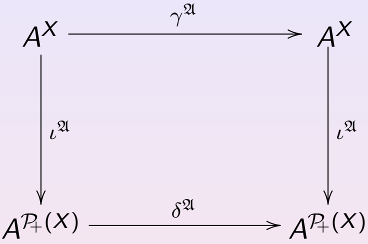


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# Regularity

### Definition

A modal $\sharp$-algebra $\mathfrak{A}$ is *regular* if

$$\iota^{\mathfrak{A}}(\mu.\gamma^{\mathfrak{A}}) \text{ is the least prefixed point of } \delta^{\mathfrak{A}}.$$

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*If $\mathfrak{A}$ is regular, then*

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# The axiomatization

We axiomatize regularity !!!

For each $\gamma \in \Gamma$, do the following:

- Out of $\gamma$, construct a system of equations $\{ \gamma_x \mid x \in X \}$, whose least solution is equivalent to the existence of the LpFP of $\gamma$.
- Construct $\{ \delta_Y \mid Y \in \mathcal{P}_+(X) \}$ so to satisfy the Lemma.
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