The Nice Labelling Problem for Event Structures

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Introduction

The problem:

- given a concurrent process,
  s.t. in each global state at most k actions may be fired,

  can it be implemented as the behavior of a
  (concurrent, deterministic) automaton
  on the alphabet $\Sigma = \{ a_1, \ldots, a_k \}$?

  If not, how large should be $\mathrm{card}(\Sigma)$?

Motivations:

- The combinatorics of models of concurrency
  as a support for algorithmic issues.
- Verification tools designed upon these models.

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Event structures

An *event structure* (with binary conflict) is a triple $\mathcal{E} = \langle E, \leq, \smile \rangle$, where

- E is a finite set of *events*,
- partially ordered by $\leq$, the *causality* relation,
- $\smile \subseteq E \times E$, the *conflict* relation, is
  - symmetric,
  - irreflexive, and
  - $x \smile y \leq z$ and implies $x \smile z$.

The (weak) *concurrency* relation:

$$x \frown y \text{ iff not } x \smile y\,.$$

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The stable domain of an event structure

For $\mathcal{E} = \langle E, \leq, \smile \rangle$ an e.s., define

$$\mathcal{I}(\mathcal{E}) = \{ I \subseteq E \mid y \leq x \in I \text{ implies } y \in I \},$$

$$\mathcal{C}(\mathcal{E}) = \{ I \in \mathcal{I}(\mathcal{E}) \mid x, y \in I \text{ implies } x \frown y \},$$

the set of (history aware) *configurations* of $\mathcal{E}$,

$$\mathcal{D}(\mathcal{E}) = \langle \mathcal{C}(\mathcal{E}), \subseteq \rangle,$$

the *domain* of $\mathcal{E}$.

The poset $\mathcal{D}(\mathcal{E})$ is a *stable domain*

– i.e. a distributive chopped lattice –

which moreover is *coherent*.

Proposition

Every stable coherent domain is of the form $\mathcal{D}(\mathcal{E})$ for some e.s. $\mathcal{E}$.

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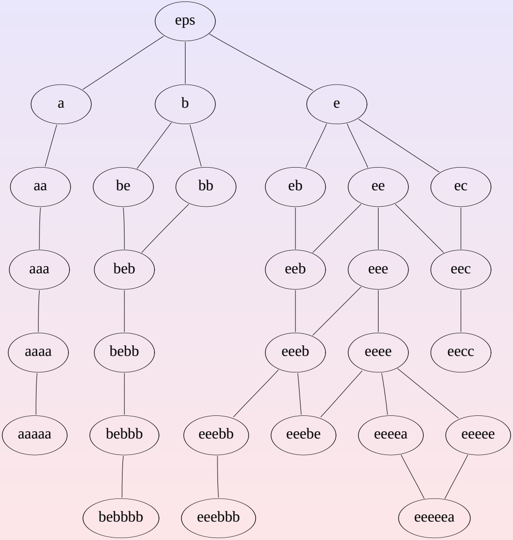
Coherent stable domain: an example



(Back to the problem . . .)

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Nice labelling:
how to transform the domain into an automaton

Hasse diagram of $\mathcal{D}(\mathcal{E})$ = state-transition graph of the process $\mathcal{E}$

Goal (definition of nice labelling of $\mathcal{E}$) :
label edges of the Hasse diagram to obtain a concurrent deterministic automaton.

- Deterministic
- Concurrent

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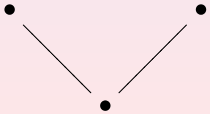

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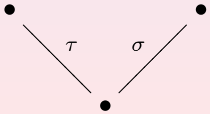

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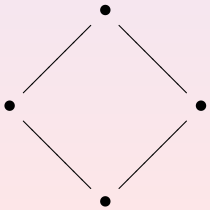

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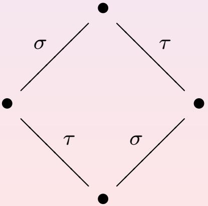

The nice labelling problem

Given an e.s. $\mathcal{E}$, compute

$$\chi(\mathcal{E}) = \min\{ \mathrm{card}(\Sigma) \mid \text{there exists a nice labelling of } \mathcal{E} \text{ into } \Sigma \}.$$

Given a class of event strucures $\mathcal{K}$, compute

$$\chi(\mathcal{K}) = \max\{ \chi(\mathcal{E}) \mid \mathcal{E} \in \mathcal{K} \}.$$

Compute $\chi(\mathcal{K}_n)$, where

$$\mathcal{K}_n = \{ \mathcal{E} \mid n \text{ is an upper bound for } \deg(\mathcal{E}) \},$$

and

$$\deg(\mathcal{E}) = \text{the maximum outdegree in the Hasse diagram of } \mathcal{D}(\mathcal{E}).$$

(Example here)

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(Example here)

Graph theoretic interpretation of the problem

For $\mathcal{E} = \langle E, \leq, \smile \rangle$, say that

$$x \bar{\frown} y \text{ iff } x, y \text{ are not comparable and} \\ x \frown y \text{ or } x, y \text{ are in minimal conflict}\,,$$

where x, y are in minimal conflict if $x \smile y$ and

$$\forall x' < x\; x' \frown y \text{ and } \forall y' < y\; x \frown y'\,.$$

Then:

$$\deg(\mathcal{E}) = \text{clique number of } \langle E, \bar{\frown} \rangle\,, \\ \chi(\mathcal{E}) = \text{chromatic number of } \langle E, \bar{\frown} \rangle\,.$$

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State of Art

Proposition (Assous, Bouchitté, Charretton, Rozoy, 1994)

$\chi(\mathcal{K}_2) = 2$, and $\chi(\mathcal{K}_n) \geq n + 1$ for $n \geq 3$.

Proposition (Dilworth, 1950)

Every finite poset can be covered with k antichains, where k is its width.

Equivalently: if $x \frown y$ forall $x, y \in E$, then $\chi(\mathcal{E}) = \deg(\mathcal{E})$.

Proposition (Assous et al., 1994)

There exists $K(n, m)$ such that if $\deg(\mathcal{E}) \leq n$ and $\mathcal{E}$ has at most m pairs in minimal conflict, then $\chi(\mathcal{E}) \leq K(n, m)$.

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Proof of Assous et al. Thm

Theorem (Assous, Bouchitté, Charretton, Rozoy)

If $\mathcal{E} \in \mathcal{K}_2$, *then* $\gamma(\mathcal{E}) \leq 2$.

The clock property:

If $x \leq y \frown z$, then $x \leq z$ or $x \frown z$.

Lemma (Antichains in degree 2)

If $x \frown y \frown z$, *then* $\{ x, z \}$ *are comparable.*

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Lemma

If a graph contains no cordless odd-length simple cycle, then it contains no odd-length simple cycle. Hence it is bipartite and can be colored with 2 colors.

Proposition

Let $\mathcal{E} = \langle E, \leq, \smile \rangle$, *with* $\deg \mathcal{E} = 2$.
Then $G = \langle E, \frown \rangle$ *contains no simple cordless cycle of length strictly greater than* 4.

Proof.

- We can divide vertices into minima and maxima
- This is a 2-coloring of the complement of a cycle
- If the length of a cycle > 4,
  then its complement contains an odd lenght cycle

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A result in degree 3

What about $\chi(\mathcal{K}_3)$?

Let us say that $\mathcal{E} = \langle E, \leq, \smile \rangle$ is *tree-like* if $\langle E, \leq \rangle$ is a tree.

Theorem

If $\mathcal{E}$ is tree-like and $\deg(\mathcal{E}) \leq 3$, then $\chi(\mathcal{E}) \leq 3$.

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Some ideas behind the proof

We say that $x, y \in E$ are *brothers* iff they have the same parents.

If $\mathcal{E}$ is tree-like, then it contains many brothers.
We can have at most $\deg(\mathcal{E})$ pairwise brothers.

Let

$$\mathcal{S}_x = \{ z \in E, x \eqsim z \mid y \nleq z \text{ if } y \text{ is a brother of } x \}.$$

Lemma

Let $\mathcal{E}$ be s.t. $\deg(\mathcal{E}) \leq 3$. If x, y are brothers, then $\mathcal{S}_x$ and $\mathcal{S}_y$ are comparable w.r.t. set inclusion and $\mathcal{S}_x \cap \mathcal{S}_y$ is a linear order.

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Ideas (contd.)

If x, y are brothers, say that

$$x \text{ is } \textit{more experienced} \text{ than } y \text{ if } \mathcal{S}_x \supset \mathcal{S}_y\,.$$

Let $\lhd$ a strict linear order on E (an age) respecting
the height and the more experienced relation bewteen brothers.

Define $\lambda(x)$ assuming it is already defined on

$$\{\, y \in E \mid y \lhd x \,\}\,,$$

taking care of

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$$\{ y \lhd x \mid y \eqsim x \} = \{ y \in \mathcal{S}_x \mid y \lhd x \} \cup \{ \text{elder brothers of } x \} .$$

The labelling

We define the coloring λ by saying that :

1. if x is an eldest brother, then x inherits the color of the father:

$$\lambda(x) = \lambda(\pi(x)) .$$

2. if x has two brothers, then $\mathcal{S}_x = \emptyset$, then

$$\{ y \lhd x \mid y \eqsim x \} = \text{brothers of } x .$$

3. if x has one (eldest and more experienced) brother y, then $\mathcal{S}_x = \mathcal{S}_x \cap \mathcal{S}_y$ has an ancestor z. We let:

$$\lambda(x) \notin \{ \lambda(y), \lambda(z) \} .$$

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Tree-like event structures in higher degree ?

Proposition (Assous et al., 1994)

There exists a family $\mathcal{E}_n$ of tree-like event structures for which

$$\chi(\mathcal{E}_n) - \deg(\mathcal{E}_n) \geq \log n - 2\,.$$

Proposition (Pouzet, L.S.)

There exists a family $\mathcal{E}_n$ of tree-like event structures for which

$$\frac{\chi(\mathcal{E}_n)}{\deg(\mathcal{E}_n)} \geq (\frac{5}{4})^{n-1}\,.$$

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Proposition (Pouzet, L.S.)

There exists a family $\mathcal{E}_n$ of tree-like event structures for which

$$\frac{\chi(\mathcal{E}_n)}{\deg(\mathcal{E}_n)} \geq (\frac{5}{4})^{n-1}\,.$$

Patching nice labellings, in degree 3

If $\deg(\mathcal{E}) = 3$, many subsets of E may be labeled with 3 colors:

- configurations:

  $$C \in \mathcal{I}(E) \text{ such that } C \text{ is a clique w.r.t. } \frown$$

  by Dilworth's Theorem.
- stars:

  $$\{ x \} \cup \{ y \mid x \,\overline{\frown}\, y \},$$

  by Rozoy' (et al.) Theorem.
- antichains, trees, ...

Sometimes this might be of help ...

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Antichains

An *antichain* of $\mathcal{E}$:

a subset A of E of pairwise uncomparable events.

If $\deg(\mathcal{E}) = 3$, then the restriction of $\frown$ to A is almost a tree:

its biconnected components are either edges or triangles.

Therefore the graph $\langle A, \frown_{|A} \rangle$ can be colored with 3 colors.

Proposition

Let

$$\mathrm{skew}(\mathcal{E}) = \max\{\, |\mathrm{hgt}(x) - \mathrm{hgt}(y)| \mid x, y \in E\,,\ x \frown y \,\}$$

If $\mathrm{skew}(\mathcal{E}) < k$, *then* $\chi(\mathcal{E}) \leq 3k$.

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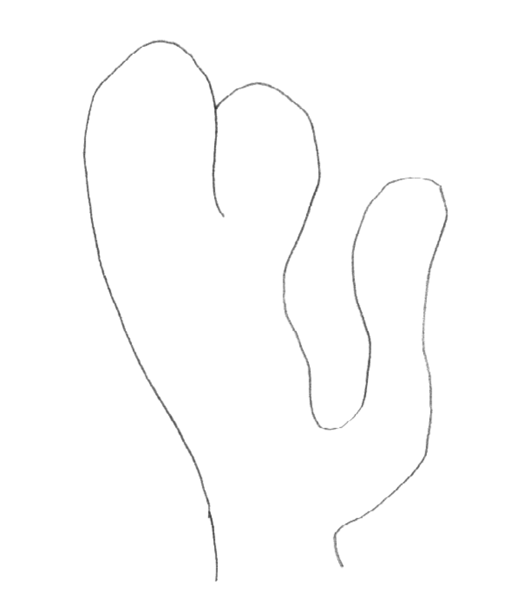
Proof idea



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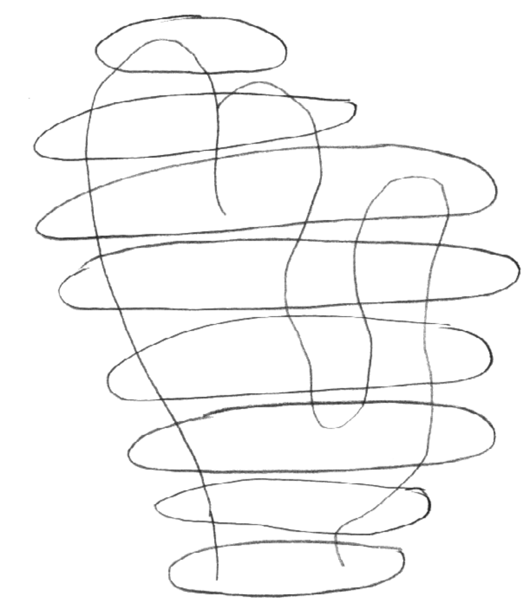

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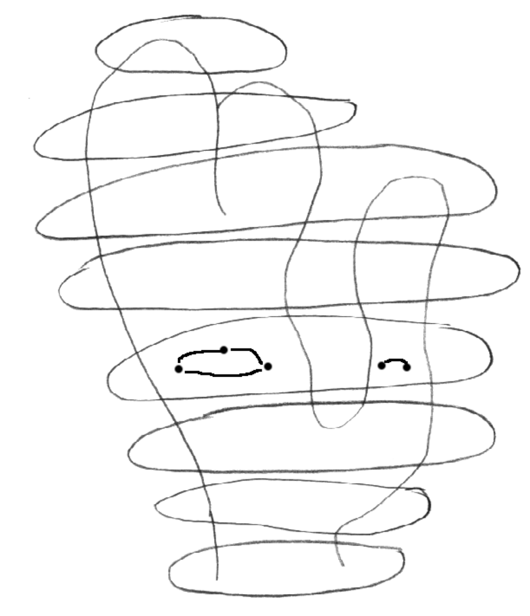

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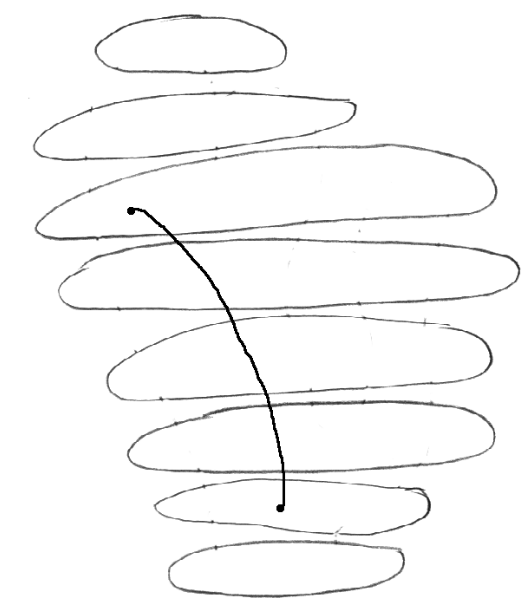

Thank you (merci)