The Nice Labelling Problem for Event Structures

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Introduction

The problem:

- given a concurrent process, s.t. in each global state at most $k$ actions may be fired,

  can it be implemented as the behavior of a (concurrent, deterministic) automaton on the alphabet $\Sigma = \{a_1, \ldots, a_k\}$?

  If not, how large should be $\text{card}(\Sigma)$?

Motivations:

- The combinatorics of models of concurrency as a support for algorithmic issues.
- Verification tools designed upon these models.
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Outline

Event Structures

The Nice Labelling Problem

Degree 2: a Proof of Assous et al. Theorem

Degree 3: Tree-like Event Structures

Perspectives
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Perspectives
Event structures

An event structure (with binary conflict) is a triple $\mathcal{E} = \langle E, \leq, \sim \rangle$, where

- $E$ is a finite set of events,
- partially ordered by $\leq$, the causality relation,
- $\sim \subseteq E \times E$, the conflict relation, is
  - symmetric,
  - irreflexive, and
  - $x \sim y \leq z$ and implies $x \sim z$.

The (weak) concurrency relation:

$x \bowtie y$ iff not $x \sim y$. 
Event structures

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The (weak) concurrency relation:

$$x \triangleright y \text{ iff not } x \triangleright y.$$
The stable domain of an event structure

For \( \mathcal{E} = \langle E, \leq, \sim \rangle \) an e.s., define

\[
\mathcal{I}(\mathcal{E}) = \{ I \subseteq E \mid y \leq x \in I \text{ implies } y \in I \},
\]

\[
\mathcal{C}(\mathcal{E}) = \{ I \in \mathcal{I}(\mathcal{E}) \mid x, y \in I \text{ implies } x \sim y \},
\]

the set of (history aware) configurations of \( \mathcal{E} \),

\[
\mathcal{D}(\mathcal{E}) = \langle \mathcal{C}(\mathcal{E}), \subseteq \rangle,
\]

the domain of \( \mathcal{E} \).

The poset \( \mathcal{D}(\mathcal{E}) \) is a stable domain

i.e. a distributive chopped lattice

which moreover is coherent.

Proposition

Every stable coherent domain is of the form \( \mathcal{D}(\mathcal{E}) \) for some e.s. \( \mathcal{E} \).
The stable domain of an event structure

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**Proposition**

*Every stable coherent domain is of the form $\mathcal{D}(\mathcal{E})$ for some e.s. $\mathcal{E}$.*
Coherent stable domain: an example
Nice labelling: how to transform the domain into an automaton

Hasse diagram of $\mathcal{D}(\mathcal{E}) = \text{state-transition graph of the process } \mathcal{E}$

Goal (definition of nice labelling of $\mathcal{E}$) :
label edges of the Hasse diagram to obtain a concurrent deterministic automaton.
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- Concurrent
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The nice labelling problem

Given an e.s. $\mathcal{E}$, compute

$$\chi(\mathcal{E}) = \min \{ \text{card}(\Sigma) \mid \text{there exists a nice labelling of } \mathcal{E} \text{ into } \Sigma \}.$$  

Given a class of event structures $\mathcal{K}$, compute

$$\chi(\mathcal{K}) = \max \{ \chi(\mathcal{E}) \mid \mathcal{E} \in \mathcal{K} \}.$$  

Compute $\chi(\mathcal{K}_n)$, where

$$\mathcal{K}_n = \{ \mathcal{E} \mid n \text{ is an upper bound for } \text{deg}(\mathcal{E}) \},$$

and

$$\text{deg}(\mathcal{E}) = \text{the maximum outdegree in the Hasse diagram of } D(\mathcal{E}).$$
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Graph theoretic interpretation of the problem

For $E = \langle E, \leq, \sim \rangle$, say that

$x \sim y$ iff $x, y$ are not comparable and

$x \preceq y$ or $x, y$ are in minimal conflict,

where $x, y$ are in minimal conflict if $x \sim y$ and

$$\forall x' < x \; x' \sim y \quad \text{and} \quad \forall y' < y \; x \sim y'.$$

Then:

$$\deg(E) = \text{clique number of } \langle E, \sim \rangle,$$

$$\chi(E) = \text{chromatic number of } \langle E, \sim \rangle.$$
Graph theoretic interpretation of the problem

For $\mathcal{E} = \langle E, \leq, \prec \rangle$, say that

$$x \preceq y \text{ iff } x, y \text{ are not comparable and }$$

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Then:

$$\deg(\mathcal{E}) = \text{clique number of } \langle E, \preceq \rangle,$$
$$\chi(\mathcal{E}) = \text{chromatic number of } \langle E, \preceq \rangle.$$
State of Art

Proposition (Assous, Bouchitté, Charretton, Rozoy, 1994)
\[ \chi(K_2) = 2, \text{ and } \chi(K_n) \geq n + 1 \text{ for } n \geq 3. \]

Proposition (Dilworth, 1950)
Every finite poset can be covered with \( k \) antichains, where \( k \) is its width.

Equivalently: if \( x \sim y \) for all \( x, y \in E \), then \( \chi(E) = \deg(E) \).

Proposition (Assous et al., 1994)
There exists \( K(n, m) \) such that if \( \deg(E) \leq n \) and \( E \) has at most \( m \) pairs in minimal conflict, then \( \chi(E) \leq K(n, m) \).
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Event Structures

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Degree 2: a Proof of Assous et al. Theorem

Degree 3: Tree-like Event Structures

Perspectives
Proof of Assous et al. Thm

Theorem (Assous, Bouchitté, Charretton, Rozoy)
If $\mathcal{E} \in \mathcal{K}_2$, then $\gamma(\mathcal{E}) \leq 2$.

The clock property:
If $x \preceq y \succeq z$, then $x \preceq z$ or $x \succeq z$.

Lemma (Antichains in degree 2)
If $x \simeq y \simeq z$, then \{ $x, z$ \} are comparable.
Proof of Assous et al. Thm

Theorem (Assous, Bouchitté, Charretton, Rozoy)
If $\mathcal{E} \in \mathcal{K}_2$, then $\gamma(\mathcal{E}) \leq 2$.

The clock property:
If $x \leq y \not\sim z$, then $x \leq z$ or $x \not\sim z$.

Lemma (Antichains in degree 2)
If $x \not\sim y \not\sim z$, then $\{x, z\}$ are comparable.
Theorem (Assous, Bouchitté, Charretton, Rozoy)
If \( \mathcal{E} \in \mathcal{K}_2 \), then \( \gamma(\mathcal{E}) \leq 2 \).

The clock property:
If \( x \triangleleft y \triangleleft z \), then \( x \leq z \) or \( x \triangleright z \).

Lemma (Antichains in degree 2)
If \( x \triangleright y \triangleright z \), then \( \{ x, z \} \) are comparable.
Lemma

If a graph contains no cordless odd-length simple cycle, then it contains no odd-length simple cycle. Hence it is bipartite and can be colored with 2 colors.

Proposition

Let $E = \langle E, \leq, \sim \rangle$, with $\deg E = 2$. Then $G = \langle E, \sim \rangle$ contains no simple cordless cycle of length strictly greater than 4.

Proof.

- We can divide vertices into minima and maxima.
- This is a 2-coloring of the complement of a cycle.
- If the length of a cycle $> 4$, then its complement contains an odd length cycle.
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If a graph contains no cordless odd-length simple cycle, then it contains no odd-length simple cycle. Hence it is bipartite and can be colored with 2 colors.

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Let $\mathcal{E} = \langle E, \leq, \bowtie \rangle$, with $\deg \mathcal{E} = 2$.
Then $G = \langle E, \bowtie \rangle$ contains no simple cordless cycle of length strictly greater than 4.

Proof.

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- If the length of a cycle $> 4$,
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Degree 2: a Proof of Assous et al. Theorem

Degree 3: Tree-like Event Structures

Perspectives
A result in degree 3

What about $\chi(K_3)$?

Let us say that $\mathcal{E} = \langle E, \leq, \triangleright \rangle$ is tree-like if $\langle E, \leq \rangle$ is a tree.

Theorem
If $\mathcal{E}$ is tree-like and $\text{deg}(\mathcal{E}) \leq 3$, then $\chi(\mathcal{E}) \leq 3$. 
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**Theorem**

If $\mathcal{E}$ is tree-like and $\deg(\mathcal{E}) \leq 3$, then $\chi(\mathcal{E}) \leq 3$. 
Some ideas behind the proof

We say that $x, y \in E$ are *brothers* iff they have the same parents.

If $E$ is tree-like, then it contains many brothers.
We can have at most $\deg(E)$ pairwise brothers.

Let

$$S_x = \{ z \in E, x \cong z \mid y \not\cong z \text{ if } y \text{ is a brother of } x \}.$$ 

**Lemma**

Let $E$ be s.t. $\deg(E) \leq 3$. If $x, y$ are brothers, then $S_x$ and $S_y$ are comparable w.r.t. set inclusion and $S_x \cap S_y$ is a linear order.
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Ideas (contd.)

If $x, y$ are brothers, say that

\[ x \text{ is more experienced than } y \text{ if } S_x \supset S_y. \]

Let $\prec$ a strict linear order on $E$ (an age) respecting the height and the more experienced relation between brothers.

Define $\lambda(x)$ assuming it is already defined on

\[ \{ y \in E \mid y \prec x \}, \]

taking care of

\[ \{ y \prec x \mid y \simeq x \}. \]
Ideas (contd.)

If $x, y$ are brothers, say that

$x$ is *more experienced* than $y$ if $S_x \supset S_y$.

Let $\triangleleft$ a strict linear order on $E$ (an age) respecting
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$$\{ y \in E \mid y \triangleleft x \} ,$$

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$$\{ y \triangleleft x \mid y \asymp x \}$$
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Let $\triangleleft$ a strict linear order on $E$ (an age) respecting the height and the more experienced relation bewteen brothers.

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$$\{ y \in E \mid y \triangleleft x \},$$

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$$\{ y \triangleleft x \mid y \approx x \}.$$
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taking care of

\[ \{ y \prec x \mid y \simeq x \} = \{ y \in S_x \mid y \prec x \} \cup \{ \text{elder brothers of } x \}. \]
We define the coloring $\lambda$ by saying that:

1. if $x$ is an eldest brother, then $x$ inherits the color of the father:

   $$\lambda(x) = \lambda(\pi(x)).$$

2. if $x$ has two brothers, then $S_x = \emptyset$, then

   $$\{ y \triangleleft x \mid y \simeq x \} = \text{brothers of } x.$$

3. if $x$ has one (eldest and more experienced) brother $y$, then $S_x = S_x \cap S_y$ has an ancestor $z$. We let:

   $$\lambda(x) \notin \{ \lambda(y), \lambda(z) \}. $$
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Perspectives
Tree-like event structures in higher degree?

Proposition (Assous et al., 1994)

There exists a family \( \mathcal{E}_n \) of tree-like event structures for which

\[
\chi(\mathcal{E}_n) - \deg(\mathcal{E}_n) \geq \log n - 2.
\]

Proposition (Pouzet, L.S.)

There exists a family \( \mathcal{E}_n \) of tree-like event structures for which

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\frac{\chi(\mathcal{E}_n)}{\deg(\mathcal{E}_n)} \geq \left( \frac{5}{4} \right)^{n-1}.
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$$\frac{\chi(\mathcal{E}_n)}{\deg(\mathcal{E}_n)} \geq \left(\frac{5}{4}\right)^{n-1}.$$
Patching nice labellings, in degree 3

If $\deg(\mathcal{E}) = 3$, many subsets of $E$ may be labeled with 3 colors:

- configurations:
  \[ C \in \mathcal{I}(E) \text{ such that } C \text{ is a clique w.r.t. } \]
  by Dilworth’s Theorem.

- stars:
  \[ \{x\} \cup \{y | x = y\} \]
  by Rozoy’ (et al.) Theorem.

- antichains, trees, . . .

Sometimes this might be of help . . .
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- stars:
  \[ \{ x \} \cup \{ y \mid x \sim y \}, \]
  by Rozoy’ (et al.) Theorem.

- antichains, trees, . . .

Sometimes this might be of help . . .
Patching nice labellings, in degree 3

If $\text{deg}(\mathcal{E}) = 3$, many subsets of $E$ may be labeled with 3 colors:

- configurations: $C \in \mathcal{I}(E)$ such that $C$ is a clique w.r.t. $\bowtie$ by Dilworth’s Theorem.

- stars: $\{x\} \cup \{y \mid x \bowtie y\}$, by Rozoy’ (et al.) Theorem.

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Antichains

An antichain of $\mathcal{E}$: a subset $A$ of $E$ of pairwise uncomparable events.

If $\text{deg}(\mathcal{E}) = 3$, then the restriction of $\preceq$ to $A$ is almost a tree: its biconnected components are either edges or triangles.

Therefore the graph $\langle A, \preceq |_A \rangle$ can be colored with 3 colors.

Proposition

Let

$$\text{skew}(\mathcal{E}) = \max \{ |\text{hgt}(x) - \text{hgt}(y)| \mid x, y \in E, x \preceq y \}$$

If $\text{skew}(\mathcal{E}) < k$, then $\chi(\mathcal{E}) \leq 3k$. 
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Proposition

Let

$$\text{skew}(\mathcal{E}) = \max\{ |\text{hgt}(x) - \text{hgt}(y)| \mid x, y \in E, x \not\preceq y \}$$

If $\text{skew}(\mathcal{E}) < k$, then $\chi(\mathcal{E}) \leq 3k$. 
Proof idea
Proof idea
Proof idea
Proof idea
Thank you (merci)