A duality for finite lattices
(The *OD*-graph of a finite lattice)

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Outline

The *OD*-graph of a finite lattice

A representation theorem

Lattice epimorphisms and P-embeddings

The general duality

Some applications

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A bit of context

Semantics/representations for (finite) lattices:

- a doubly ordered set $(X, \leq_1, \leq_2)$ — Urquhart
- a table $(J, \leq, M)$ — Wille, FCA, Gehrke, Hartonas, Dunn

Goal:

"propose" a semantics framework

- attractive for modal logicians and co-algebraists
- useful in practice (for the working lattice theorist)

- an *OD*-graph $(J, \leq, \mathcal{M})$

Nation, Freese, Ježek, Wehrung, Semenova, Grätzer,
Bertet, Caspard, Monjardet,
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The *OD*-graph of a lattice L

Definition

The *OD*-graph of a (finite) lattice L is the structure $(J(L), \leq, \mathcal{M})$ where

- $(J(L), \leq)$ is the restriction of the order to join-irreducible els,
- for each $j \in J(L)$

$$\mathcal{M}(j) = \{ \, C \mid C \text{ is a minimal join-cover of } j \, \}$$

and

$$O = \text{Order}$$
$$D = \text{Depencency relation between join-irreducible els}$$

defined by

$$jDk \text{ iff } j \neq k \text{ and } \exists C \in \mathcal{M}(j) \text{ s.t. } k \in C\,.$$

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Join-covers in a lattice

L a lattice, $J(L)$ set of join-irreducible els. For $C, D \subseteq J(L)$ let

$$C \ll D \quad \text{iff} \;\; \forall c \in C \; \exists d \in D \text{ s.t. } c \leq d\,.$$

For $j \in J(L)$ and $C \subseteq J(L)$, say that

- C is a join-cover of j if

$$j \leq \bigvee C$$

- C is a minimal join-cover of j if $j \leq \bigvee C$, and

$$j \leq \bigvee D \;\; \text{and } D \ll C \quad \text{implies} \quad C \subseteq D$$

- C is a trivial join-cover of j if $\{\, j \,\} \ll C$.

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The standard representation

Theorem

Say that a downset $X \subseteq J(L)$ is closed if

$$C \subseteq X \text{ and } C \in \mathcal{M}(j) \text{ implies } j \in X$$

and let

$$\mathfrak{L}(J(L), \leq, \mathcal{M}) = \text{ lattice of closed subsets of } J(L)\,.$$

Then

$$L \simeq \mathfrak{L}(J(L), \leq, \mathcal{M})\ .$$

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An example

A lattice L is *n-distributive*, that is, it satifies

$$x \wedge \Big(\bigvee_{i=0,\ldots,n} y_i \Big) = \bigvee_{i=0,\ldots,n} \Big(x \wedge \bigvee_{j \neq i} y_j \Big)$$

if and only if

$$\#(C) \leq n$$

for each $j \in J(L)$ and $C \in \mathcal{M}(j)$.

A lattice L is *distributive* iff

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Characterizing *OD*-graphs

Theorem

A relational structure $(J, \leq, M)$ is isomorphic to an OD-graph of some lattice iff $(J, \leq)$ is a poset and, for each $j \in J$, the following hold:

1. $\{ j \} \in M(j)$
2. C *is an antichain w.r.t.* $\leq$ – *for each* $C \in M(j)$
3. $M(j)$ *is an antichain w.r.t.* $\ll$
4. $C \in M(j)$ *and* $C \ll \{ k \}$ *implies* $j \leq k$
5. *if* $C \in M(j)$ *and* $D_c \in M(c)$, $c \in C$, *then* *there exists* $E \in M(j)$ *such that* $E \ll \bigcup_{c \in C} D_c$

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Lattice epimorphisms and P-embeddings

- If $f : L \to M$ is a lattice morphism, then $\ell \dashv f$, with $\ell : M \to L$.
- f is an epimorphism iff ℓ is monic (an embedding) and moreover

$$\ell_{|J(M)} : J(M) \longrightarrow J(L) .$$

Theorem

If f is an epimorphism, then the restriction $\ell_{|J(M)}$ is a P-morphism:

$$C \in \mathcal{M}(\ell(j)) \quad \textit{iff} \quad \textit{there exists } D \in \mathcal{M}(j)\, s.t.\, \ell(D) = C .$$

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$\mathfrak{Latt}_{\text{fin,epi}}$: category of finite lattices and epimorphisms
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Theorem
The contravariant functor

$$(J(\_), \leq, \mathcal{M}) : \mathfrak{Latt}_{\text{fin,epi}} \longrightarrow OD\text{-}\mathfrak{Gr}_{\text{pe}}$$

is full and faithful.
Its pseudo-inverse is given by the closure lattice of an OD-graph:

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Directed bisimulations

Let $X = (X, \leq, M)$, $Y = (Y, \leq, M)$ be two OD-graphs.

Definition

A relation $L \subseteq X \times Y$ is a directed bisimulation if

- L is an downset of $X^{op} \times Y$
- xL is closed, for each $x \in X$
- $\forall x \in X \exists C \in M(x)$ s.t. $\bigcup_{c \in C} cL \subseteq xL$
- xLy and $C \in M(x)$ implies
  there exists $D \in M(y)$ s.t. $\bigcup_{c \in C} cL \supseteq D$,
- $x\bar{L}y$ and $D \in M(y)$ implies
  there exists $C \in M(x)$ s.t. $C \subseteq \bigcup_{d \in D} \bar{L}d$

where

$$x\bar{L}y \text{ iff } y \geq z \text{ for all } z \text{ s.t. } xLz\,.$$

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The lattice of closed subsets of an OD-graph

$$\mathfrak{L}(\_) : OD\text{-}\mathfrak{Gr} \longrightarrow \mathfrak{Latt}_{\mathsf{fin}}$$

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The representation of arrows

An OD-graph $(X, \leq, M)$:

a presentation of a join-semilattice:

$$j_X : \mathcal{O}(X, \leq) \longrightarrow \mathcal{O}(X, \leq)/j_X = \mathfrak{L}(X, \leq, M)$$

with j_X is the closure operator induced by M.

A simulation from $(X, \leq, M)$ to $(Y, \leq, M)$:

a join-homomorphism F s.t. $Ker(j_X) \subseteq Ker(f)$:

The representation of arrows

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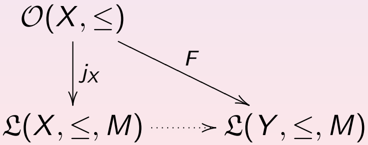

Simulations

Lemma

Let $X = (X, \leq, M)$ and $Y = (Y, \leq, M)$. There is a posets-iso

$$\mathfrak{Latt}_{\mathsf{fin},\vee}(\mathfrak{L}(X), \mathfrak{L}(Y)) \simeq OD\text{-}\mathfrak{Gr}_{\mathsf{sim}}(X, Y)$$

where $OD\text{-}\mathfrak{Gr}_{\mathsf{sim}}(X, Y)$ is the set of simulations $R \subseteq X \times Y$, i.e.:

- *R is a lower set of $X^{op} \times Y$*
- *xR is closed, for each $x \in X$*
- *R is a simulation:*

$$xRy \text{ and } C \in M(x) \text{ implies}$$

$$\text{there exists } D \in M(y) \text{ s.t. } \bigcup_{c \in C} cR \supseteq D\,.$$

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Left adjoints in $OD\text{-}\mathfrak{Gr}_{\mathsf{sim}}$

$$\begin{aligned}\mathfrak{Latt}_{\mathsf{fin}}(X, Y) &= \{\text{ right adjoints in } \mathfrak{Latt}_{\mathsf{fin},\vee}(X, Y)\,\} \\ &= \{\text{ left adjoints in } \mathfrak{Latt}_{\mathsf{fin},\vee}(Y, X)\,\}\end{aligned}$$

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Lemma

$L : X \longrightarrow Y$ is a left adjoint in $OD\text{-}\mathfrak{Gr}_{\mathsf{sim}}$, that is, there exists $R : Y \longrightarrow X$ such that

$$L \fatsemi R \subseteq Id_X\,, \qquad Id_Y \subseteq R \fatsemi L\,,$$

iff L is a directed bisimulation.

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Outline

A Kripke (cover) semantics for lattice terms

Let $(J, \leq, M)$ be an OD-graph and $v : X \to P(J)$ be closed.

$$
\begin{aligned}
j \models x \quad &\text{iff} \quad j \in v(x) \\
j \models \bigwedge_{i \in I} t_i \quad &\text{iff} \quad j \models t_i\,, \quad \forall i \in I \\
j \models \bigvee_{i \in I} t_i \quad &\text{iff} \quad \exists C \in M(j) \text{ s.t. } \forall c \in C \exists i \in I \text{ s.t. } c \models t_i
\end{aligned}
$$

For a lattice terms t, we have

$$j \models t \quad \text{iff} \quad j \in \tilde{v}(t)\,.$$

Extremal fixed-points on finite lattices

Context, the μ-terms:

$$t = x \mid \bigwedge_{i \in I} t_i \mid \bigvee_{i \in I} t_i \mid \mu_x.t \mid \nu_x.t \, .$$

Theorem

For each n, there is a variety of lattices $\mathcal{D}_n$ such that, for any lattice polynomial ϕ, the equation

$$\phi^{n+1}(\bot) = \phi^n(\bot)$$

holds on its finite members.

Moreover, on $\mathcal{D}_{3n}$, we have that

$$\phi^n(\bot) \neq \phi^{n-1}(\bot) \, .$$

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Thank you for the attention,

. . . questions?