A duality for finite lattices
(The $OD$-graph of a finite lattice)

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Outline

The $OD$-graph of a finite lattice

A representation theorem

Lattice epimorphisms and P-embeddings

The general duality

Some applications
Outline

The \textit{OD}-graph of a finite lattice

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Lattice epimorphisms and P-embeddings

The general duality

Some applications
A bit of context

Semantics/representations for (finite) lattices:

▶ a doubly ordered set \((X, \leq_1, \leq_2)\)  
  Urquhart

▶ a table \((J, \leq, M)\)  
  Wille, FCA, Gehrke, Hartonas, Dunn

Goal:

“propose” a semantics framework

- attractive for modal logicians and co-algebraists
- useful in practice (for the working lattice theorist)

▶ an OD-graph \((J, \leq, M)\)

Nation, Freese, Ježek, Wehrung, Semenova, Grätzer, 
Bertet, Caspard, Monjardet, 
... Rob Goldblatt
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The *OD*-graph of a lattice $L$

**Definition**
The *OD*-graph of a (finite) lattice $L$ is the structure $(J(L), \leq, \mathcal{M})$ where

- $(J(L), \leq)$ is the restriction of the order to join-irreducible els,
- for each $j \in J(L)$ \[ \mathcal{M}(j) = \{ C \mid C \text{ is a minimal join-cover of } j \} \]

and

- $O = \text{Order}$
- $D = \text{Dependency relation between join-irreducible els}$

defined by

\[ jDk \text{ iff } j \neq k \text{ and } \exists C \in \mathcal{M}(j) \text{ s.t. } k \in C. \]
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Join-covers in a lattice

$L$ a lattice, $J(L)$ set of join-irreducible els. For $C, D \subseteq J(L)$ let

$$C \ll D \quad \text{iff} \quad \forall c \in C \exists d \in D \text{ s.t. } c \leq d.$$ 

For $j \in J(L)$ and $C \subseteq J(L)$, say that

- $C$ is a join-cover of $j$ if $j \leq \bigvee C$,
- $C$ is a minimal join-cover of $j$ if $j \leq \bigvee C$, and $j \leq \bigvee D$ and $D \ll C$ implies $C \subseteq D$,
- $C$ is a trivial join-cover of $j$ if $\{j\} \ll C$. 
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The standard representation

Theorem
Say that a downset $X \subseteq J(L)$ is closed if

$$C \subseteq X \text{ and } C \in \mathcal{M}(j) \text{ implies } j \in X$$

and let

$$\mathcal{L}(J(L), \leq, \mathcal{M}) = \text{ lattice of closed subsets of } J(L).$$

Then

$$L \simeq \mathcal{L}(J(L), \leq, \mathcal{M}).$$
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An example

A lattice $L$ is *n-distributive*, that is, it satisfies

$$x \land ( \bigvee_{i=0,\ldots,n} y_i ) = \bigvee_{i=0,\ldots,n} (x \land \bigvee_{j \neq i} y_j)$$

if and only if

$$\#(C) \leq n$$

for each $j \in J(L)$ and $C \in \mathcal{M}(j)$.

A lattice $L$ is *distributive* iff

$$\mathcal{M}(j) = \{ \{ j \} \}.$$
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The \textit{OD}-graph of a finite lattice

A representation theorem

Lattice epimorphisms and P-embeddings

The general duality

Some applications
Characterizing $OD$-graphs

**Theorem**

A relational structure $(J, \leq, M)$ is isomorphic to an $OD$-graph of some lattice iff $(J, \leq)$ is a poset and, for each $j \in J$, the following hold:

1. $\{ j \} \in M(j)$
2. $C$ is an antichain w.r.t. $\leq$ – for each $C \in M(j)$
3. $M(j)$ is an antichain w.r.t. $\ll$
4. $C \in M(j)$ and $C \ll \{ k \}$ implies $j \leq k$
5. if $C \in M(j)$ and $D_c \in M(c)$, $c \in C$, then there exists $E \in M(j)$ such that $E \ll \bigcup_{c \in C} D_c$
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- The $OD$-graph of a finite lattice
- A representation theorem
- Lattice epimorphisms and $P$-embeddings
- The general duality
- Some applications
Lattice epimorphisms and P-embeddings

- If \( f : L \to M \) is a lattice morphism, then \( \ell \dashv f \), with \( \ell : M \to L \).

- \( f \) is an epimorphism iff \( \ell \) is monic (an embedding) and moreover
  \[
  \ell_{|J(M)} : J(M) \longrightarrow J(L).
  \]

Theorem

If \( f \) is an epimorphism, then the restriction \( \ell_{|J(M)} \) is a P-morphism:

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C \in M(\ell(j)) \text{ iff there exists } D \in M(j) \text{ s.t. } \ell(D) = C.
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A first duality

\( \text{Latt}_{\text{fin, epi}} : \) category of finite lattices and epimorphisms

\( \text{OD-Gr}_{\text{pe}} : \) category of OD-graphs and \( P \)-embeddings

Theorem

The contravariant functor

\[
(J(\_), \leq, M) : \text{Latt}_{\text{fin, epi}} \longrightarrow \text{OD-Gr}_{\text{pe}}
\]

is full and faithful.

Its pseudo-inverse is given by the closure lattice of an OD-graph:

\[
\mathcal{L}(\_) : \text{OD-Gr}_{\text{pe}} \longrightarrow \text{Latt}_{\text{fin, epi}}.
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\( \mathbb{L} \text{att}_{\text{fin, epi}} : \) category of finite lattices and epimorphisms

\( \mathcal{O}D\mathcal{G} \text{r}_{\text{pe}} : \) category of \( \mathcal{O}D \)-graphs and \( P \)-embeddings

**Theorem**

*The contravariant functor*

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(J(-), \leq, \mathcal{M}) : \mathbb{L} \text{att}_{\text{fin, epi}} \longrightarrow \mathcal{O}D\mathcal{G} \text{r}_{\text{pe}}
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The $OD$-graph of a finite lattice

A representation theorem

Lattice epimorphisms and $P$-embeddings

The general duality

Some applications
Directed bisimulations

Let \( X = (X, \leq, M) \), \( Y = (Y, \leq, M) \) be two OD-graphs.

**Definition**

A relation \( L \subseteq X \times Y \) is a directed bisimulation if

- \( L \) is an downset of \( X^{op} \times Y \)
- \( xL \) is closed, for each \( x \in X \)
- \( \forall x \in X \exists C \in M(x) \text{ s.t. } \bigcup_{c \in C} cL \subseteq xL \)
- \( xLy \) and \( D \in M(y) \) implies there exists \( C \in M(x) \) s.t. \( C \subseteq \bigcup_{d \in D} \bar{L}d \)

where

\( x \bar{L} y \) iff \( y \geq z \) for all \( z \) s.t. \( xLz \).
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Back 15/23
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The general duality (representation theorem)

\( \mathbf{Latt}_{\text{fin}} \): the category of finite lattices

\( \mathbf{OD-Gr} \): category of OD-graphs and directed bisimulations

**Theorem**

The contravariant functor

\[ (J(\_), \leq, M) : \mathbf{Latt}_{\text{fin}} \rightarrow \mathbf{OD-Gr} \]

is full and faithful.

The lattice of closed subsets of an OD-graph

\[ \mathcal{L}(\_): \mathbf{OD-Gr} \rightarrow \mathbf{Latt}_{\text{fin}} \]

gives rise to a contravariant pseudo-inverse.
The general duality (representation theorem)

\( \mathcal{L}_{\text{fin}} : \) the category of finite lattices

\( OD-\mathcal{Gr} : \) category of \( OD \)-graphs and directed bisimulations

**Theorem**

The contravariant functor
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The general duality (representation theorem)

\( \mathbb{Latt}_{\text{fin}} \): the category of finite lattices

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The representation of arrows

An $OD$-graph $(X, \leq, M)$:

a presentation of a join-semilattice:

$$j_X : \mathcal{O}(X, \leq) \longrightarrow \mathcal{O}(X, \leq)/j_X = \mathcal{L}(X, \leq, M)$$

with $j_X$ is the closure operator induced by $M$.

A simulation from $(X, \leq, M)$ to $(Y, \leq, M)$:

a join-homomorphism $F$ s.t. $\text{Ker}(j_X) \subseteq \text{Ker}(f)$:

$$\begin{array}{ccc}
\mathcal{O}(X, \leq) & \xrightarrow{j_X} & \mathcal{L}(X, \leq, M) \\
\downarrow & & \downarrow \\
\mathcal{L}(X, \leq, M) & \xrightarrow{F} & \mathcal{L}(Y, \leq, M)
\end{array}$$
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\downarrow j_X \\
\mathcal{L}(X, \leq, M) \\
\end{array}
\xrightarrow{F}
\begin{array}{c}
\mathcal{L}(Y, \leq, M)
\end{array}
\]
Lemma

Let $X = (X, \leq, M)$ and $Y = (Y, \leq, M)$. There is a posets-iso

$$\mathbb{L}_{\text{fin}, \lor}(\mathcal{L}(X), \mathcal{L}(Y)) \simeq \text{OD-Gr}_{\text{sim}}(X, Y)$$

where $\text{OD-Gr}_{\text{sim}}(X, Y)$ is the set of simulations $R \subseteq X \times Y$, i.e.:

- $R$ is a lower set of $X^{\text{op}} \times Y$
- $xR$ is closed, for each $x \in X$
- $R$ is a simulation:

  $xRy$ and $C \in M(x)$ implies there exists $D \in M(y)$ s.t. $\bigcup_{c \in C} cR \supseteq D$. 
Simulations

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Lemma

Let \( X = (X, \leq, M) \) and \( Y = (Y, \leq, M) \). There is a posets-iso

\[ \mathbb{L}_{\text{fin}, \vee}(\mathcal{L}(X), \mathcal{L}(Y)) \cong OD-\mathcal{G}_{\text{sim}}(X, Y) \]

where \( OD-\mathcal{G}_{\text{sim}}(X, Y) \) is the set of simulations \( R \subseteq X \times Y \), i.e.:

- \( R \) is a lower set of \( X^{\text{op}} \times Y \)
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Left adjoints in $OD$-$Gr_{\text{sim}}$

\[ \text{Latt}_{\text{fin}}(X, Y) = \{ \text{right adjoints in Latt}_{\text{fin}, \lor}(X, Y) \} \]
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\[ \text{Latt}_{\text{fin}}^{\text{op}}(Y, X) = \{ \ell \in \text{Latt}_{\text{fin}, \lor}(X, Y) \mid \ell \text{ is a left adjoint} \} \]
\[ = \{ L \in OD$-$Gr_{\text{sim}}(X, Y) \mid L \text{ is a left adjoint} \} \].

Lemma
$L : X \rightarrow Y$ is a left adjoint in $OD$-$Gr_{\text{sim}}$, that is, there exists $R : Y \rightarrow X$ such that

\[ L \supseteq R \subseteq \text{Id}_X, \quad \text{Id}_Y \subseteq R \supseteq L, \]

iff $L$ is a directed bisimulation.
Left adjoints in $OD$-$Gr_{sim}$

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Outline

The $OD$-graph of a finite lattice

A representation theorem

Lattice epimorphisms and P-embeddings

The general duality

Some applications
A Kripke (cover) semantics for lattice terms

Let $(J, \leq, M)$ be an $OD$-graph and $\nu : X \rightarrow P(J)$ be closed.

\[
\begin{align*}
j \models x & \iff j \in \nu(x) \\
j \models \bigwedge_{i \in I} t_i & \iff j \models t_i, \ \forall i \in I \\
j \models \bigvee_{i \in I} t_i & \iff \exists C \in M(j) \text{ s.t. } \forall c \in C \exists i \in I \text{ s.t. } c \models t_i
\end{align*}
\]

For a lattice terms $t$, we have

\[
j \models t \iff j \in \tilde{\nu}(t).
\]
Extremal fixed-points on finite lattices

Context, the $\mu$-terms:

$$ t = x \mid \bigwedge_{i \in I} t_i \mid \bigvee_{i \in I} t_i \mid \mu_x.t \mid \nu_x.t. $$

Theorem

For each $n$, there is a variety of lattices $\mathcal{D}_n$ such that, for any lattice polynomial $\phi$, the equation

$$ \phi^{n+1}(\bot) = \phi^n(\bot) $$

holds on its finite members.

Moreover, on $\mathcal{D}_{3n}$, we have that

$$ \phi^n(\bot) \neq \phi^{n-1}(\bot). $$
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Thank you for the attention, ... questions?