Causal graph dynamics
—Internship topics—
Reversibility, stochasticity or geometry.

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Abstract

Cellular Automata consist in a grid of identical cells, each of which may take a state among a finite set. The state of each cell at time $t+1$ is computed by applying a fixed local rule to the state of cell at time t, and that of its neighbours, synchronously and homogeneously across space. Causal Graph Dynamics [1] extend Cellular Automata from fixed grids, to time-varying graphs. That is, the whole graph evolves in discrete time steps, and this global evolution is required to have a number of physics-like symmetries: *shift-invariance* (it acts everywhere the same), *causality* (information has a bounded speed of propagation).

One could also introduce any of these three other ingredients:

- *Reversibility* (the whole evolution is a bijection over the set of labelled graphs). That way we obtain a Computer Science-inspired toy model, that has features mimicking both Quantum theory (reversibility) and General Relativity (evolving topology).
- *A geometrical interpretation* (by making sure that the graphs are dual to simplicial complexes, i.e. triangles glued together). That way we obtain an evolution of discrete surfaces, mimicking General Relativity even more closely.
- *Sochasticity* (by introducing noise, non-determinism). That way we can seek to model phenomena such as social networks, protein-folding and other kinds of reconfigurable networks within the model.

The student will choose between any of these three subtopics according to taste, and be encouraged to tackle some concrete questions we have begun to address.

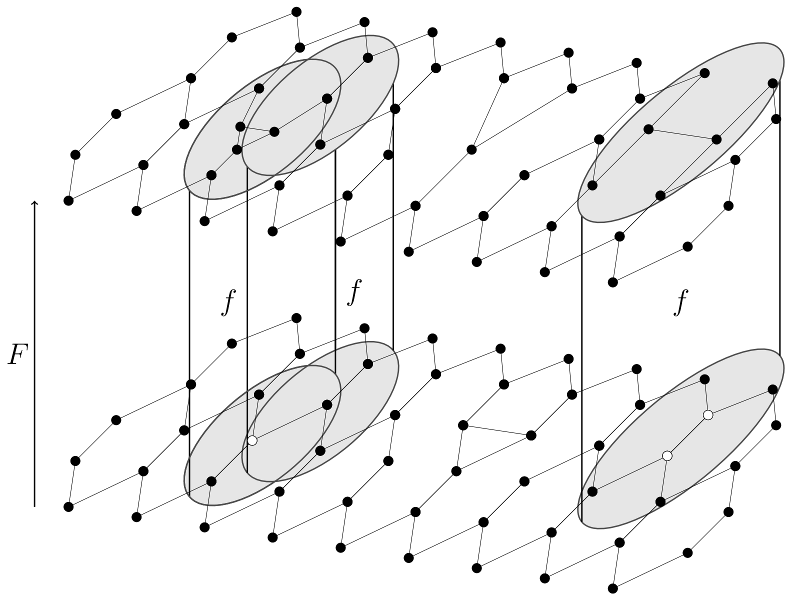

Figure 1: *Informal illustration of Causal Graph Dynamics.* The entire graph evolves into another according to a global function F. But this evolution is causal (information propagates at a bounded speed) and homogeneous (same causes lead to same effects). This has been shown to be equivalent to applying a local function f to every subdisk of the input graphs, leading to small output graphs whose union make up the output graph.

1 Background

Cellular Automata (CA) consist in a $\mathbb{Z}^n$ grid of identical cells, each of which may take a state among a finite set Σ. Thus the configurations are in $\Sigma^{\mathbb{Z}^n}$. The state of each cell at time $t+1$ is given by applying a fixed local rule f to the cell and its neighbours, synchronously and homogeneously across space. CA constitute the most established model of computation that accounts for euclidean space. They are widely used to model spatially distributed computation (self-replicating machines, synchronization problems…), as well as a great variety of multi-agents phenomena (traffic jams, demographics…). But their origin lies in Physics, where they are commonly used to model waves or particles. And since small scale physics is understood to be reversible, it was natural to endow them with this further, physics-like symmetry: reversibility.
Reversible Cellular Automata (RCA) The study of Reversible CA (RCA) was further motivated by the promise of lower energy consumption in reversible computation. RCA have turned out to have a beautiful mathematical theory, which relies on topological and algebraic characterizations in order to prove that the inverse of a CA is a CA, or in order to prove that any RCA can be expressed as a finite-depth circuits of local reversible permutations or 'blocks'.
Causal Graph Dynamics (CGD) [1, 4, 2], on the other hand, deal with a twofold extension of CA. First, the underlying grid is extended to being an arbitrary – possibly infinite – bounded-degree graph G. Informally, this means that each vertex of the graph may take a state among a finite set Σ, so a configuration is an element of $\Sigma^{V(G)}$, and the edges of the graph stand for the locality of the evolution: the next state of a vertex depends only on the states of the vertices which are at distance at most k, i.e. in a disk of radius k, for some fixed integer k. Second, the graph itself is allowed to evolve over time. Informally, this means having configurations in a set composed of the union of $\Sigma^{V(G)}$ for all possible bounded-degree G: $\bigcup_G \Sigma^{V(G)}$. This has led to a model where the local rule f is applied synchronously and homogeneously on every possible subdisk of the input graph, thereby producing small patches of the output graphs, whose union constitutes the output graph. Figure 1 illustrates the concept of these CA over graphs.
CGD are motivated by the countless situations in which some agents interact with their neighbours, leading to a global dynamics in which the notion of who is next to whom also varies in time (e.g. agents become physically connected, get to exchange contact details, move around…). Indeed, several existing models (of physical systems, computer processes, biochemical agents, economical agents, social networks…) feature such neighbour-to-neighbour interactions with time-varying neighbourhood, thereby generalizing CA for their specific sake (e.g. self-reproduction, discrete general relativity à la Regge calculus, etc. CGD provide a theoretical framework, for these models.
Reversible Causal Graph Dynamics (RCGD) are but CGD in the reversible regime. Specific examples of these were described by Hasslacher, Meyer, Love. From a theoretical Computer Science perspective, the point is therefore to generalize RCA theory to time-varying graphs; this is what we did in [6, 5]. From a theoretical physics perspective, the question whether the reversibility of small scale physics (quantum mechanics, micro-mechanical), can be reconciled with the time-varying topology of large scale physics (relativity), is a topic of debate and constant investigation. This object provides a toy, discrete, classical model where reversibility and time-varying topology coexist and interact.

2 Concrete questions

Questions on Reversibility. We have already proven that any RCGD can be expressed as a finite-depth circuit of reversible gates (i.e. local permutations of subdisks of the graph). These gates commute with one another. If we want to use this result to build some concrete examples of RCGD, we need to be able to generate reversible gates that commute with one another. This is hard in general, but easy if we make sure that act according to some partitions, so that at any given stage, their actions do not overlap. We would like to prove that these Partitioned CGD are universal within RCGD.
We would also like to prove that grid-like graphs can emerge from RCGD. This would be of particular interest to theoretical physicists, who are always looking for ideas that could explain the emergence of euclidean space as we know it.
Questions on a geometrical interpretation. In order to discretize a surface, we can break it into

triangles, glued together along their sides. This is a simplicial complex. We can then build the dual graph: each triangle is a node, and a glueing is an edge. That way, our graph evolutions can be though of as discrete surface evolutions. We did this in the 2D case in [3]. This can be extended in 3D by gueing tetrahedra instead. Some beautiful questions arise, such as characterizing discretized manifolds within those graphs, and the dynamics that preserve this property. We have a number of things worked out already, but not all.

Questions on Sochasticity The aim is to use CGD to model phenomena such as social networks, protein-folding and other kinds of real-life reconfigurable networks. For this we need a noisy, stochastic extention of CGD. We have worked out the definitions and properties of stochastic CGD already. But we still lack a good example.

3 Internship environment

The internship will take place either in the CaNa group of LIF, Marseille or at the IXXI in ENS-Lyon (the second option is subject to office space being available). The CaNa research group (Pablo Arrighi, Giuseppe Di Molfetta, Kevin Perrot, Sylvain Sen) seeks to capture at the formal level some of the fundamental paradigms of theoretical physics and biology, via the models and approaches of theoretical computer science and discrete mathematics. The group is located in Luminy, Marseille, France, and benefits from a rich scientific environment with the Cellular Automata experts of I2M (Pierre Guillon, Guillaume Theyssier) and the physicists from CPT (Alberto Vega, Thomas Krajewski).

Do not hesitate to get in touch.

References

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