A survey on transducers

decidability, logic and algebra

Pierre-Alain Reynier

LIF, Aix-Marseille University & CNRS

Formal methods to improve software

Software systems are complex and ubiquitous

- critical systems ➜ reliability
- widespread ➜ efficiency, scalability

➜ need for formal methods

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Automata-based approaches:

- model checking
- controller synthesis
- performance evaluation
- model optimization

Objective: Improve our theoretical understanding of automata models

From Languages to Transductions

Languages	Transductions
$Input \to \{0,1\}$	$Input \to Outputs$
automata	transducers
accept inputs	transform inputs

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Applications:

- Word-to-word transducers:
  - language and speech processing
  - model-checking infinite state-space systems
  - reactive systems
  - verification of web sanitizers (tool BEK)
  - string pattern matching
- Nested-word-to-word transducers:
  - XML transformations
  - model for recursive programs

Two-fold objective: theory and applications

Develop the theory of transducers/transductions:

- expressiveness
- closure properties
- decidability

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and their applications:

- verification
  - equivalence, type checking ($T(L_{in}) \subseteq L_{out}$)
- model optimization
  - streaming evaluation, minimization of resources

Simplification of models

General Problem

Given a (complex) model of a transformation, does there exist an equivalent simpler model?

➜ Natural question:

- minimization of automata
- make model deterministic
- reduce number of registers
- 2way: reduce number of passes
- ...

Automata/Logic/Algebra connections

Rich theory of regular languages: multiple presentation

Each presentation owns its assets:

- Automata: decision problems
- Logic: closure properties
- Algebra: characterizations of subclasses

➜ try to lift this theory to transductions

Overview

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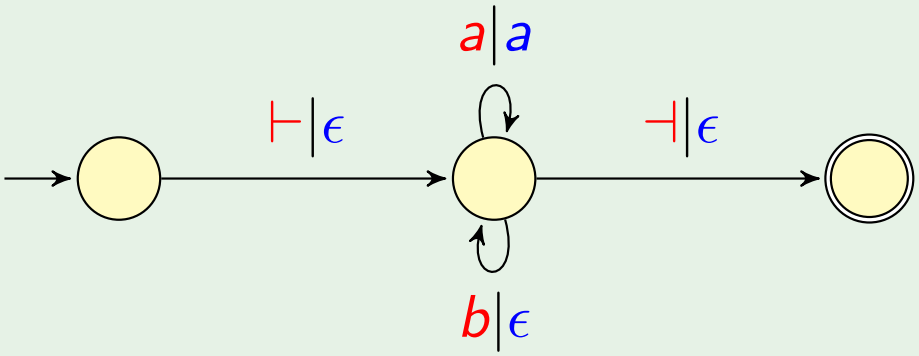
(Two-way) finite state transducers

= associate output words with transitions of a finite state automaton

Example (A transducer T)



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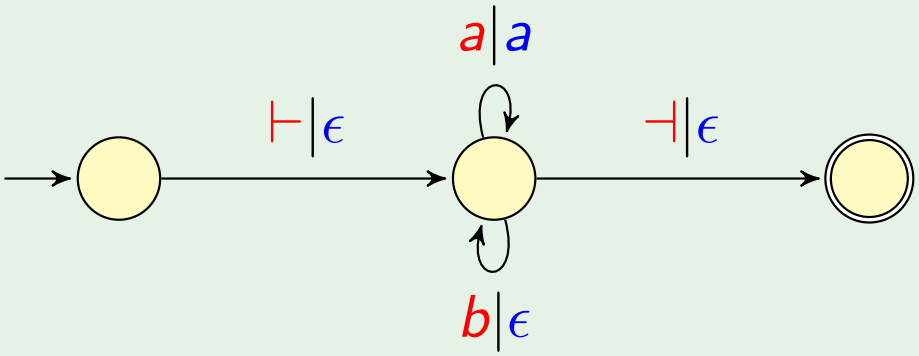

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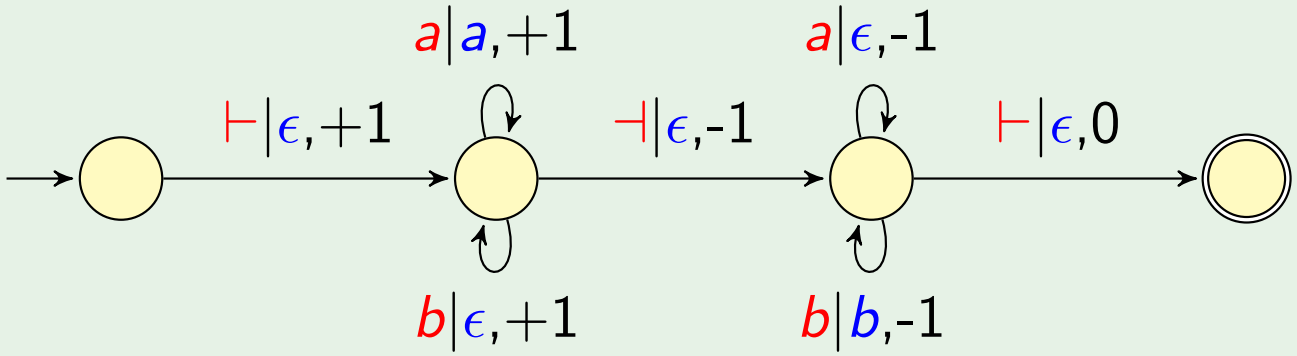

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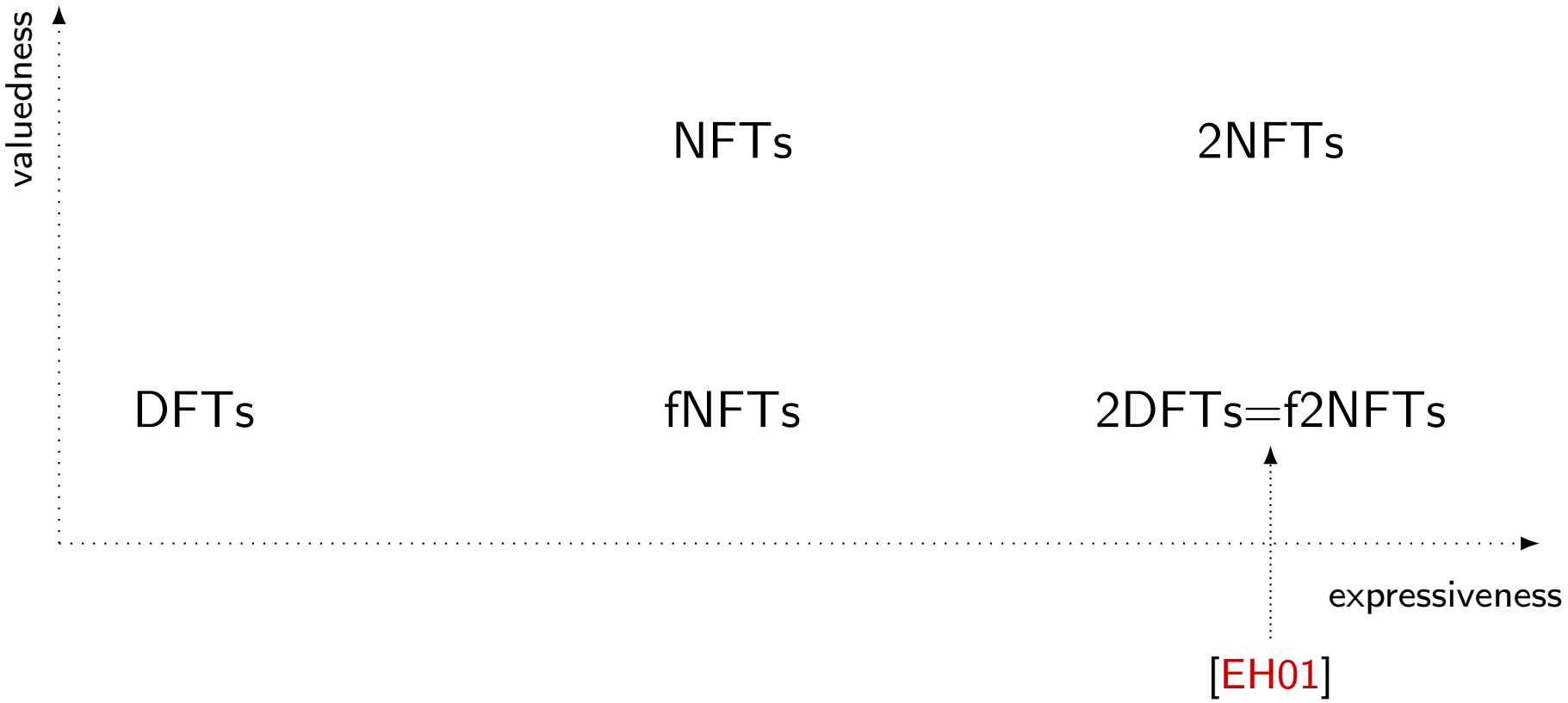
Classes of Transductions



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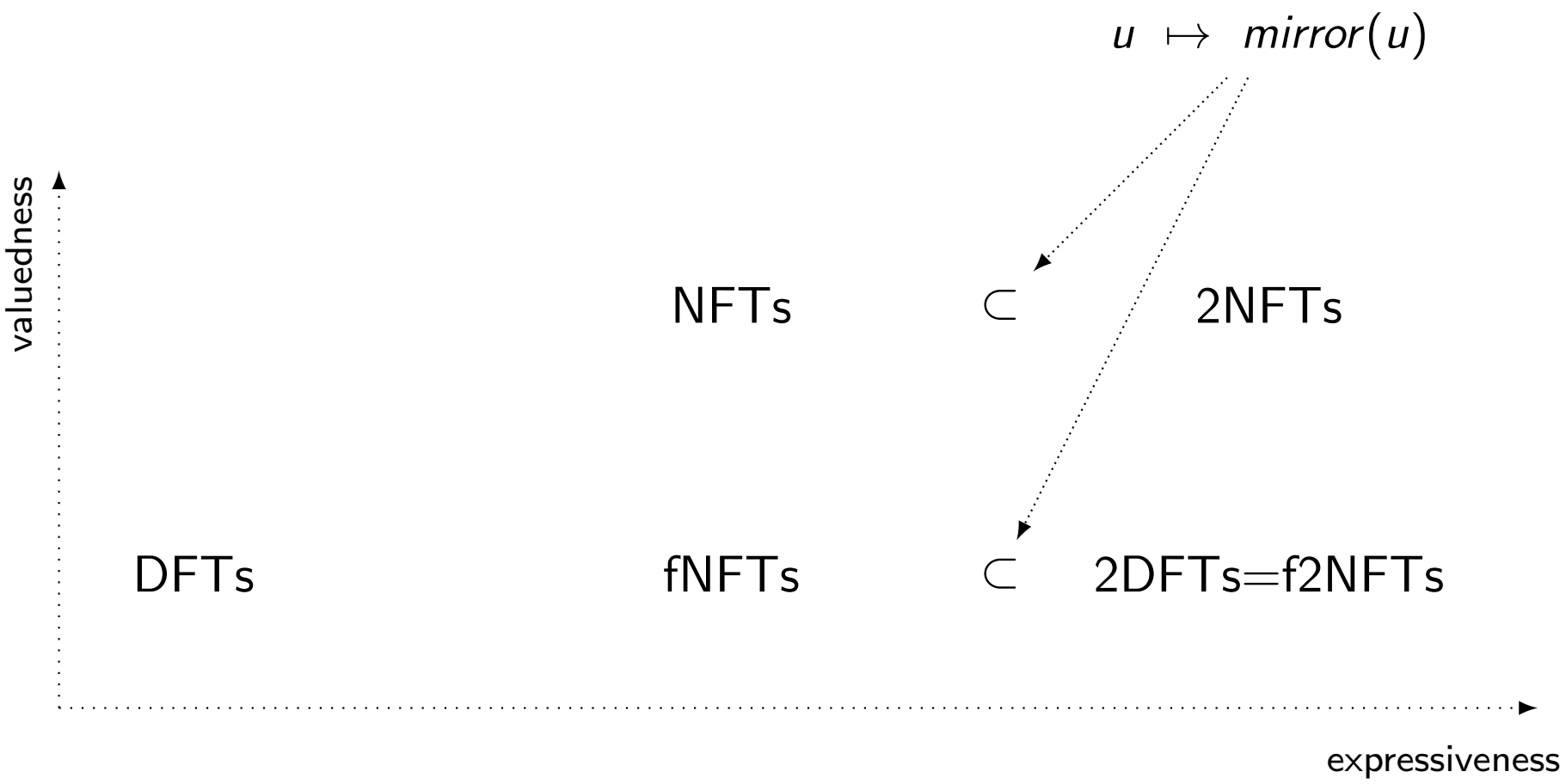

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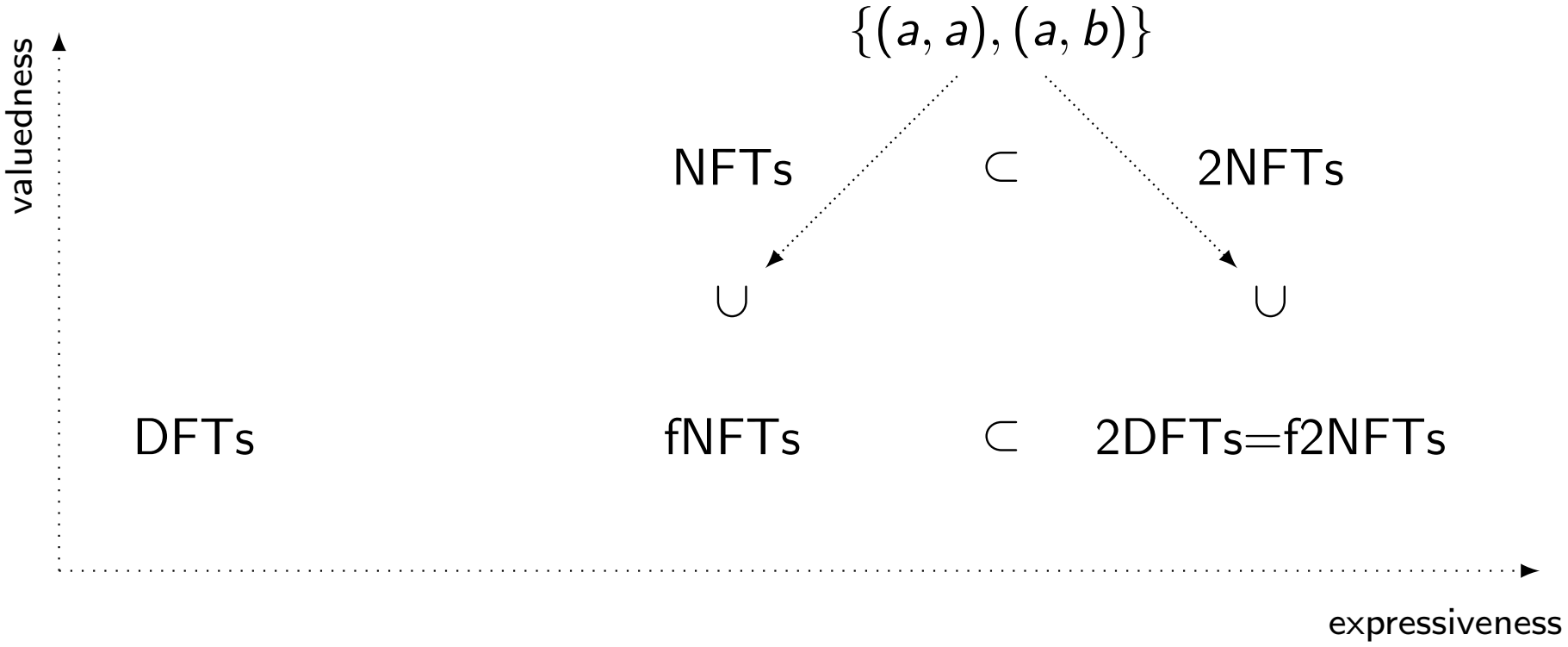

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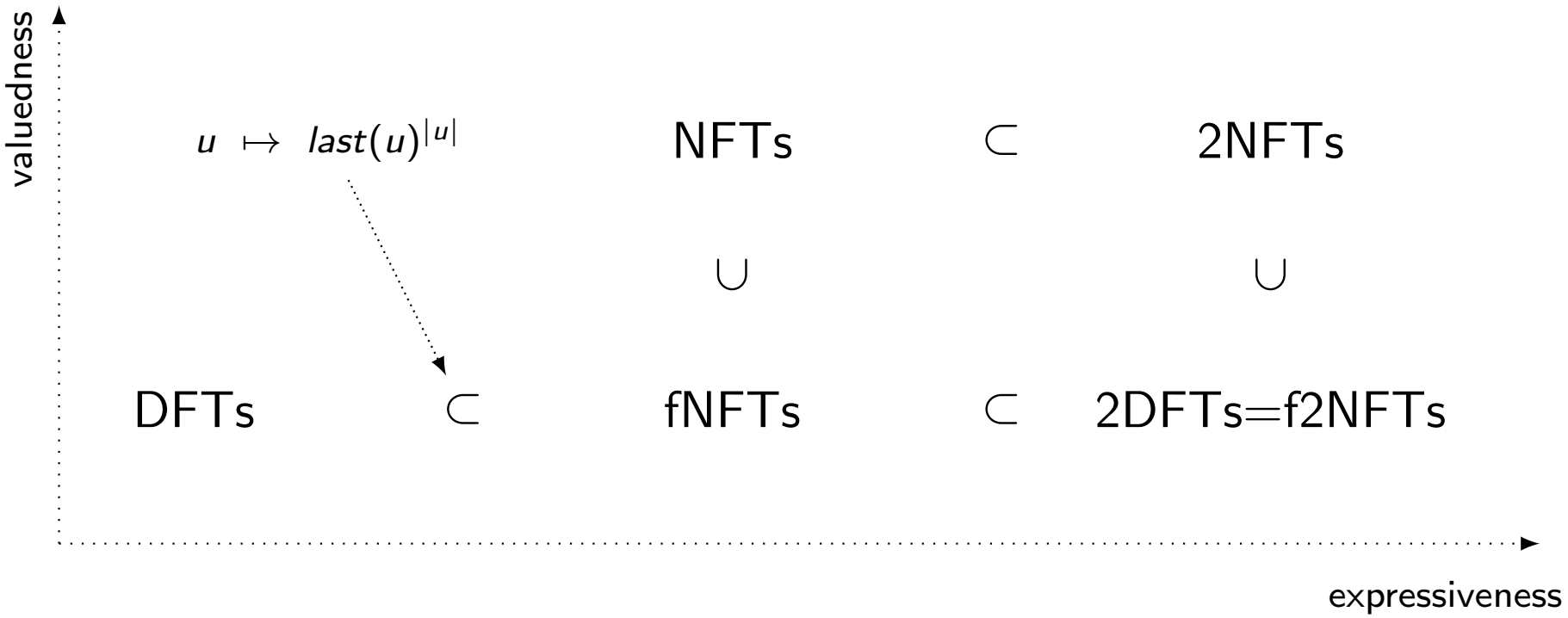

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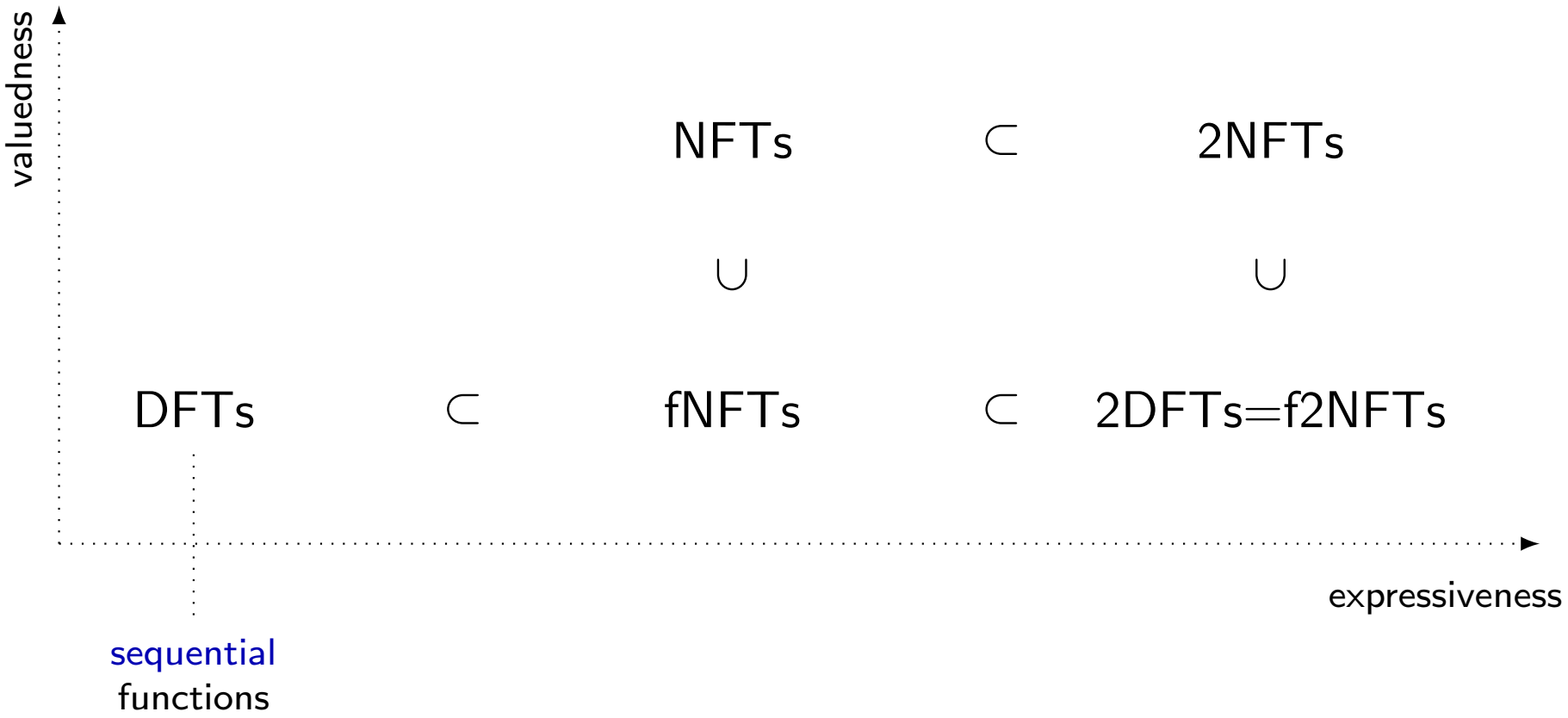

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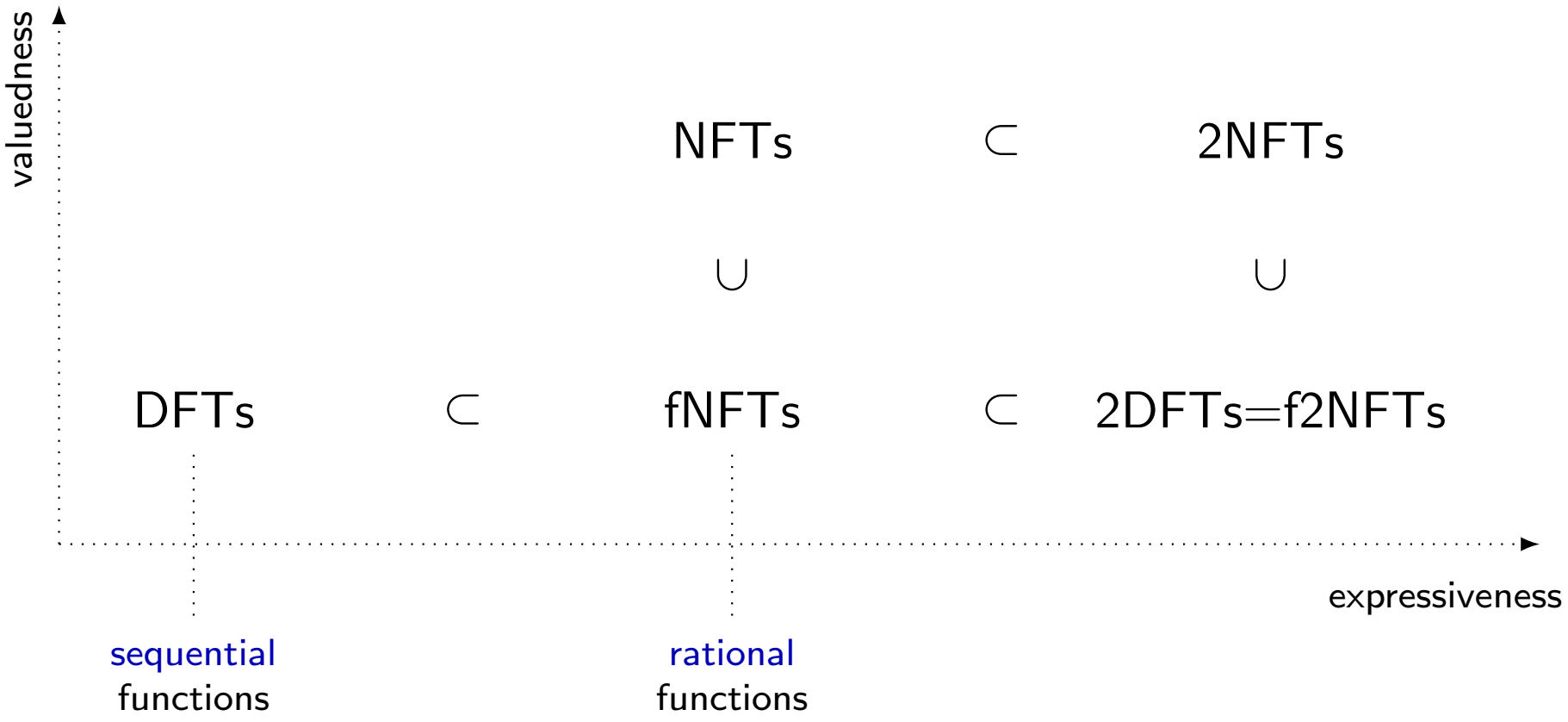

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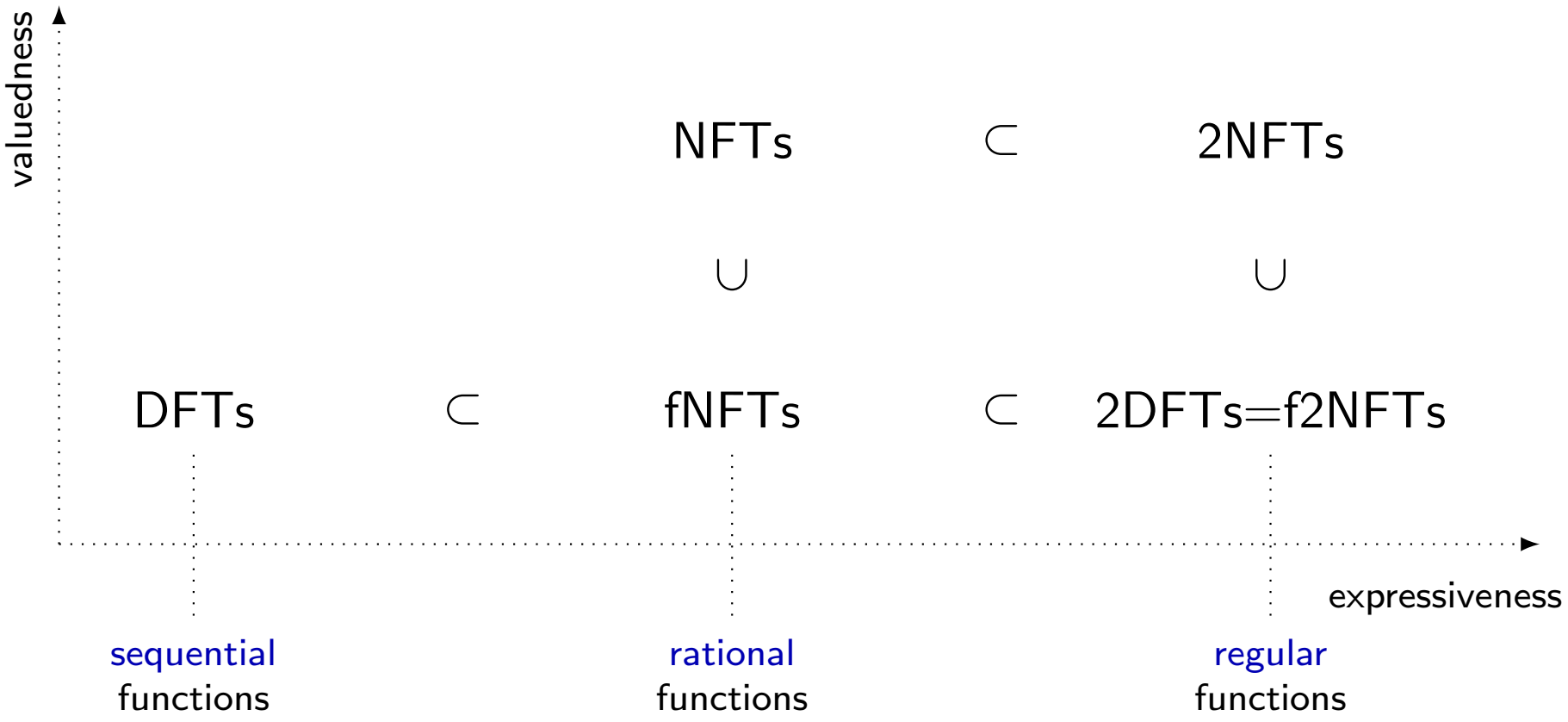

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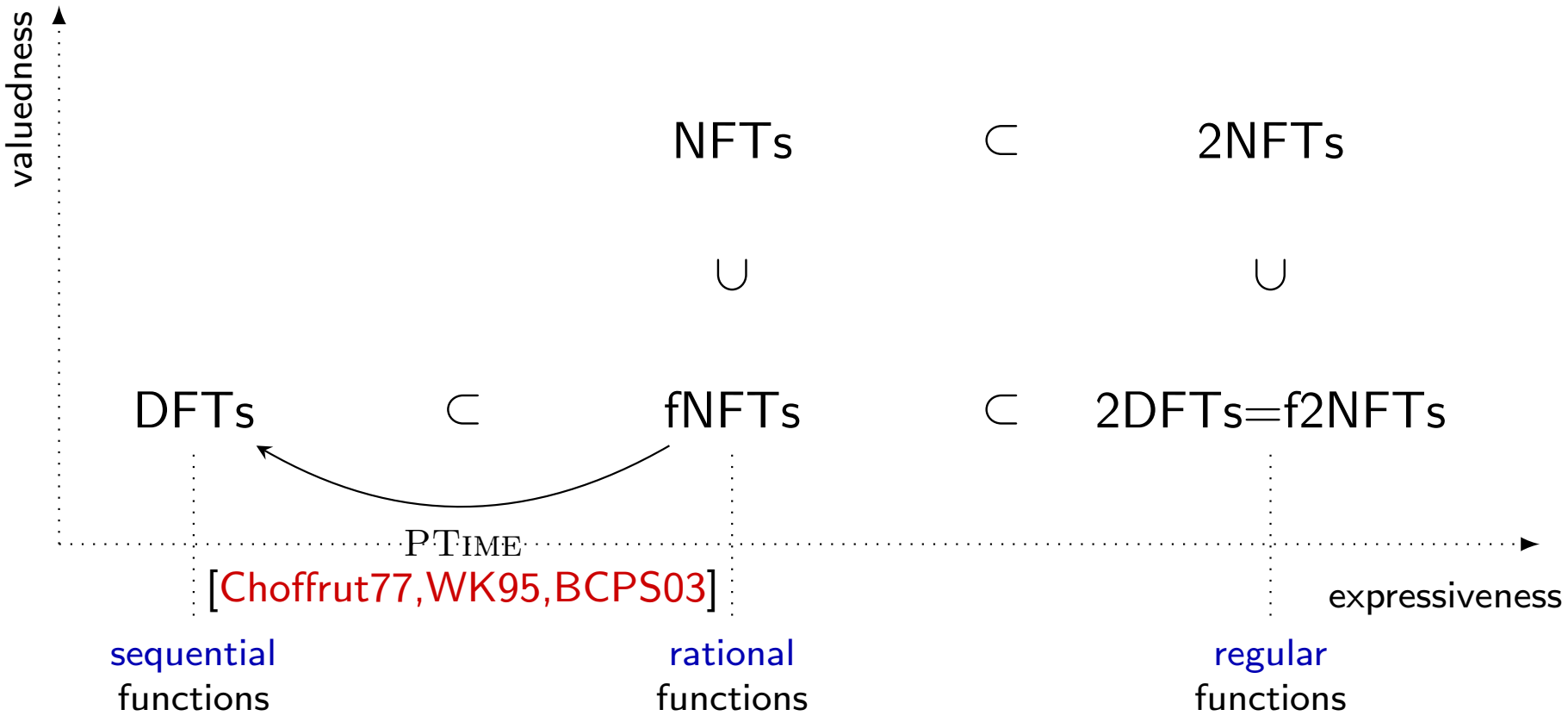

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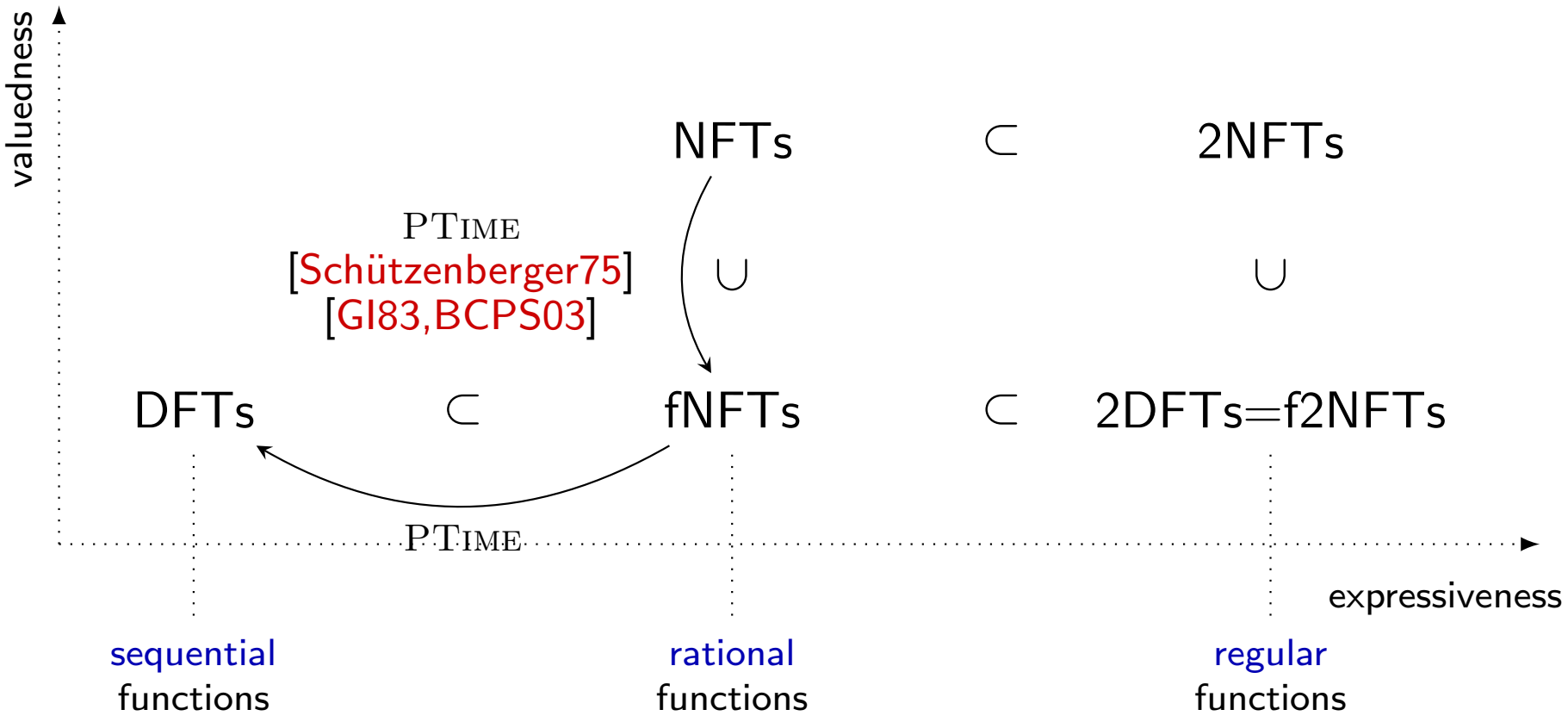

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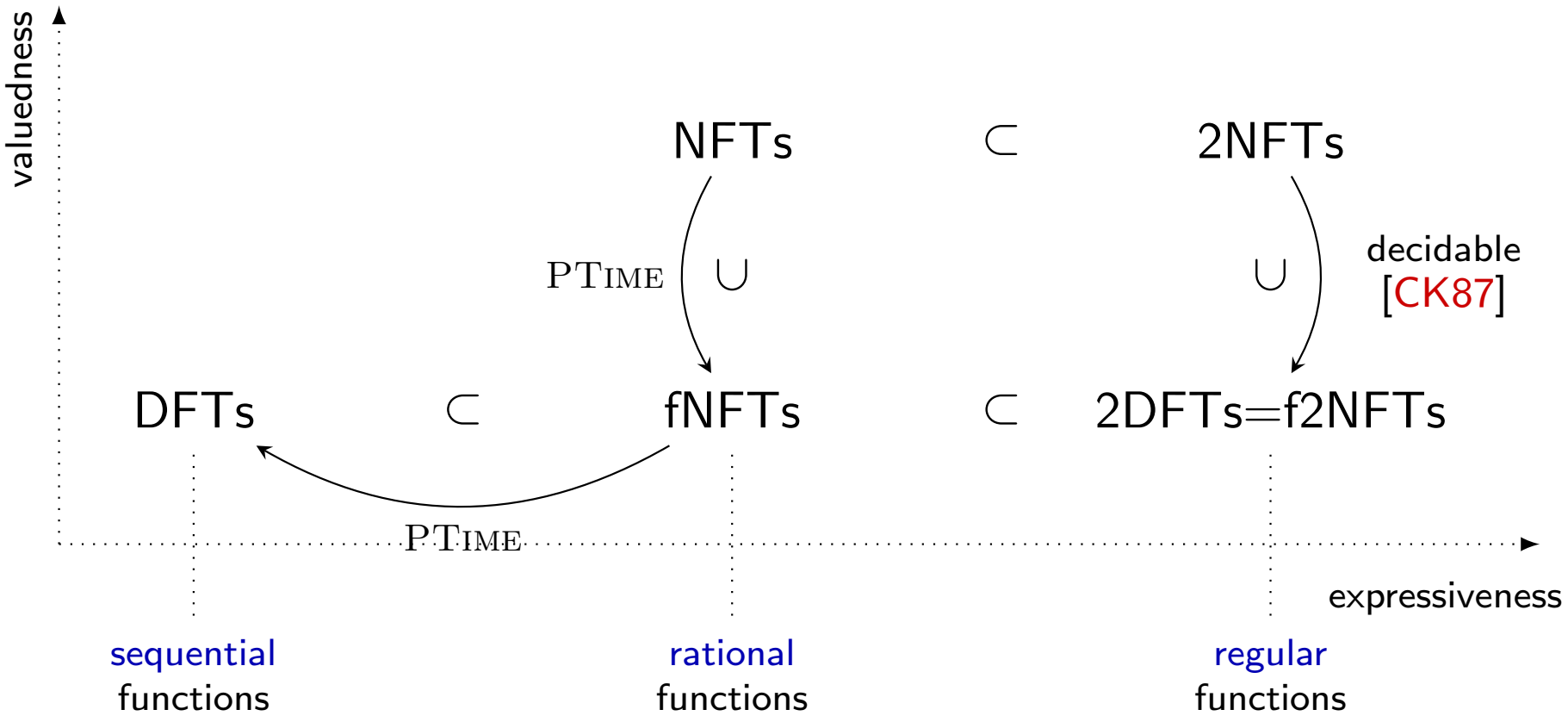

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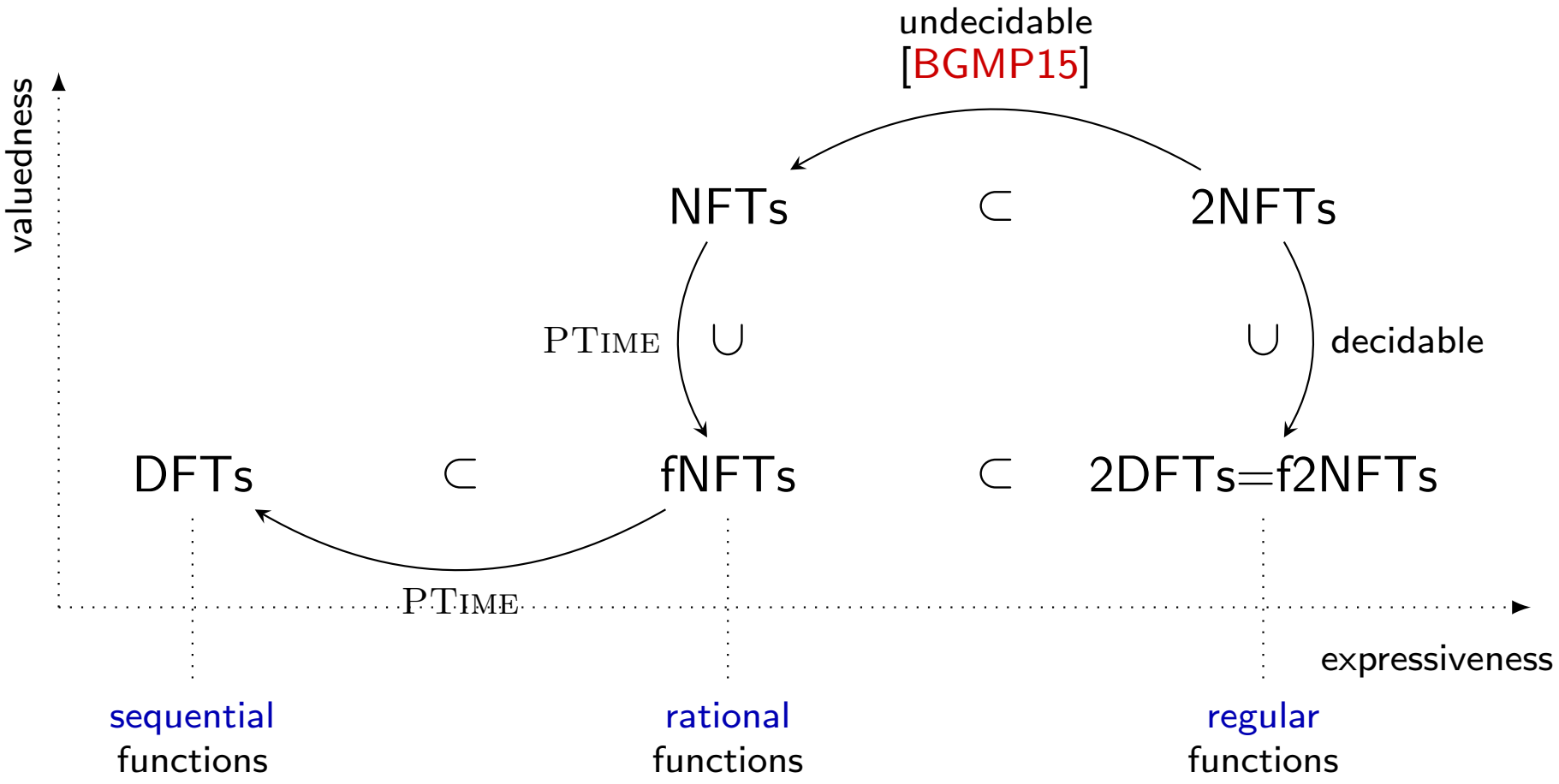

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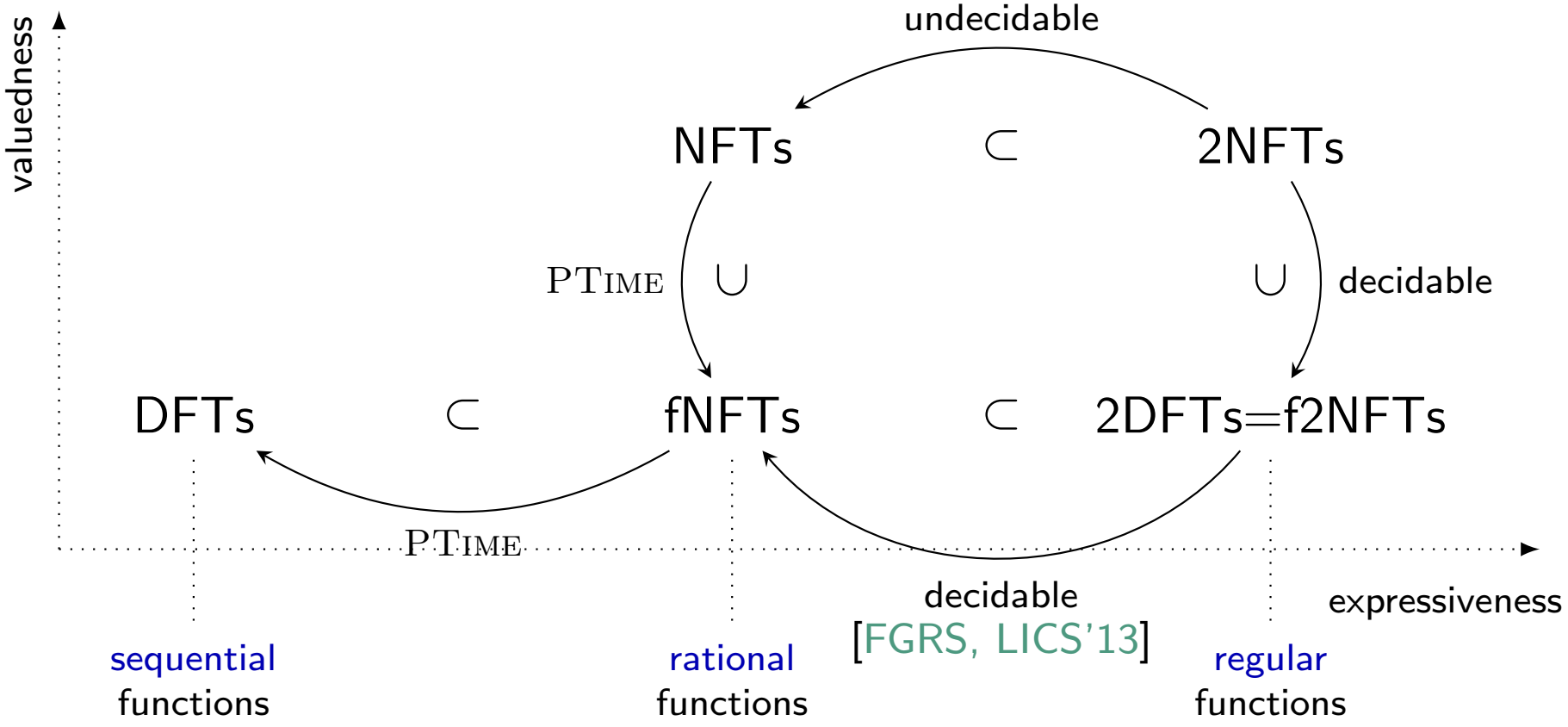


Streaming String Transducers [AC10]

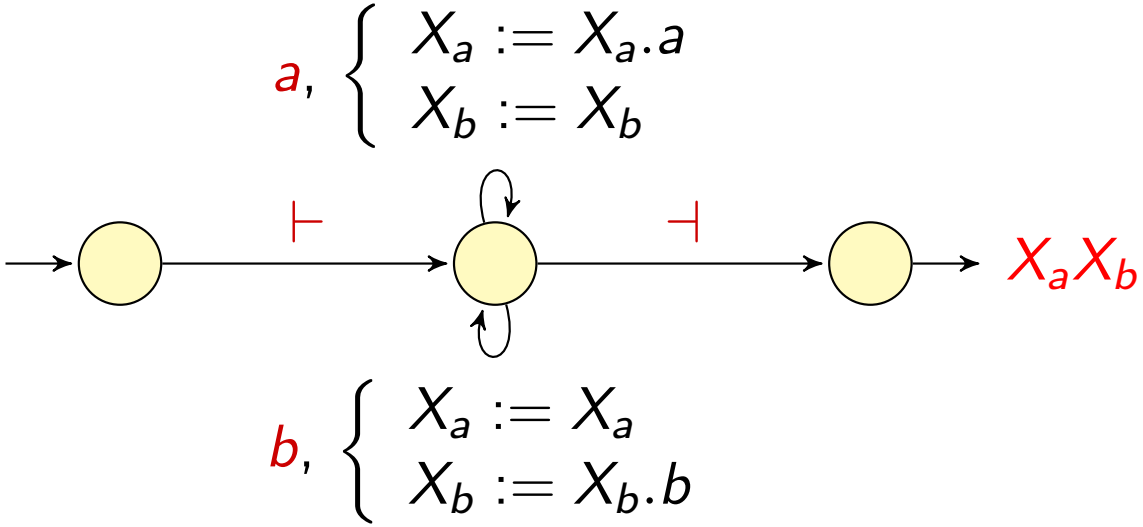
$\vdash w \dashv \;\mapsto\; a^{\#_a(w)} b^{\#_b(w)}$

1-Way DFA + registers

Register updates:

- $X := u \cdot Y \cdot v$
- $X := YZ$

X, Y, Z: registers
u, v: words in Σ^*



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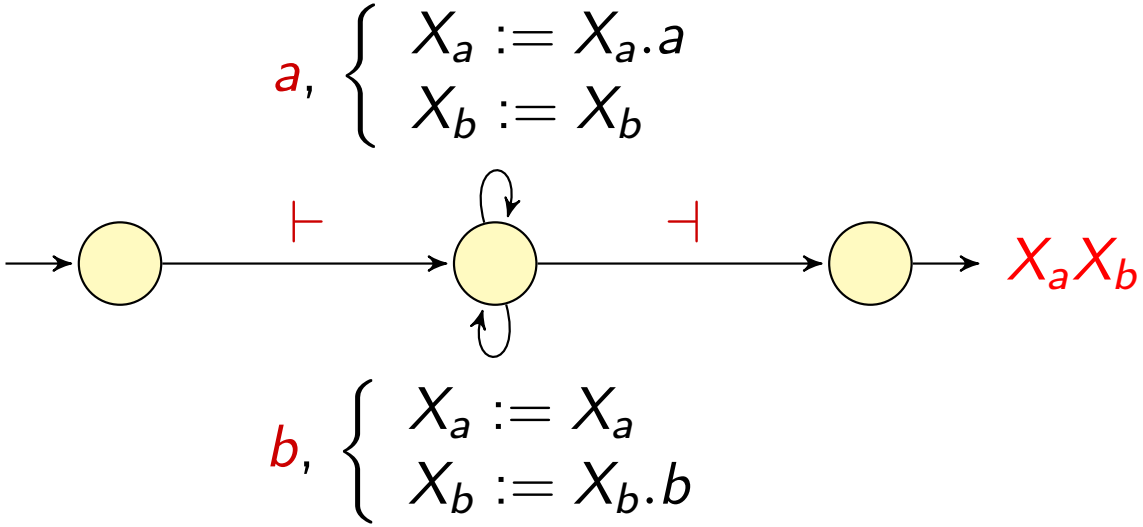

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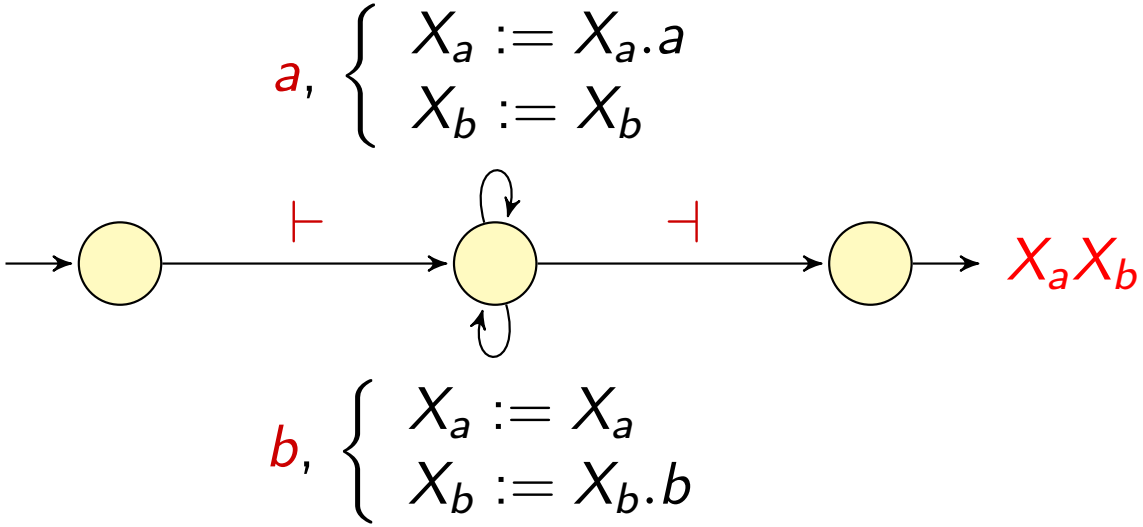

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- rational functions: right appending SST $X := Y.a$

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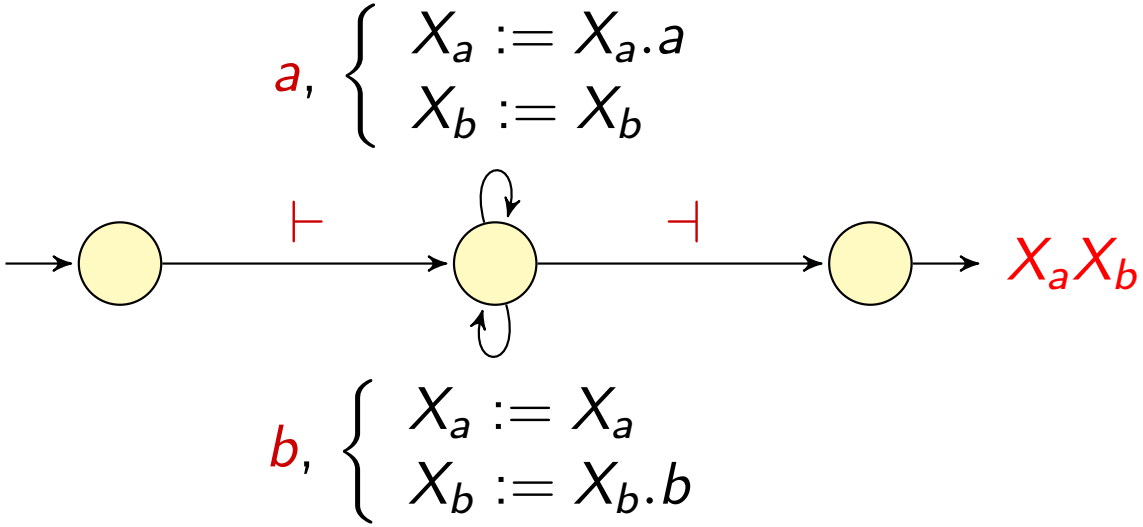


Expressiveness results :

- sequential functions: 1-register right appending SST $X := X.a$
- rational functions: right appending SST $X := Y.a$
- regular functions: copyless SST (X,Y):=(X,X) is forbidden

Overview

From rational to sequential functions [Choffrut77]

Sequentiality Problem

Input: a fNFT
Question: does there exist an equivalent DFT?

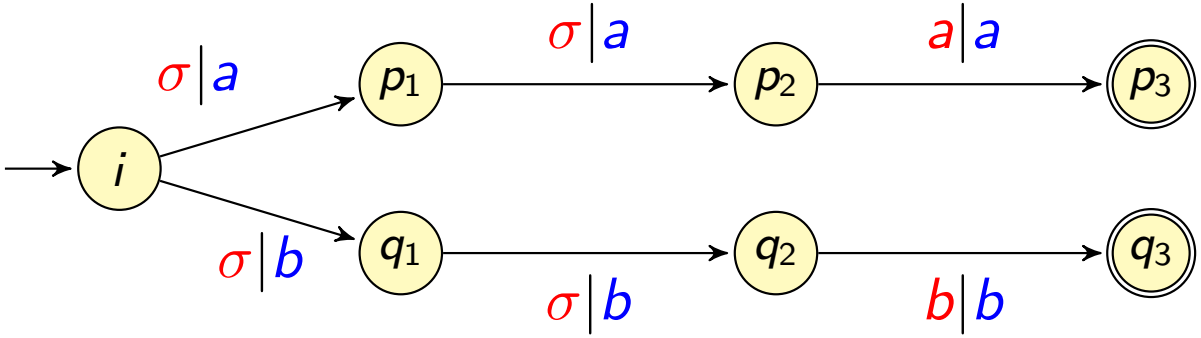
Standard technique:

- subset construction starting from the set of initial states.
- output longest common prefix
- store the unproduced outputs in the state

States of the form $\{(p, a), (q, \varepsilon), (s, bb)\}$

An example

$dom(f) = \Sigma^3$
$f(u) = last(u)^{|u|}$



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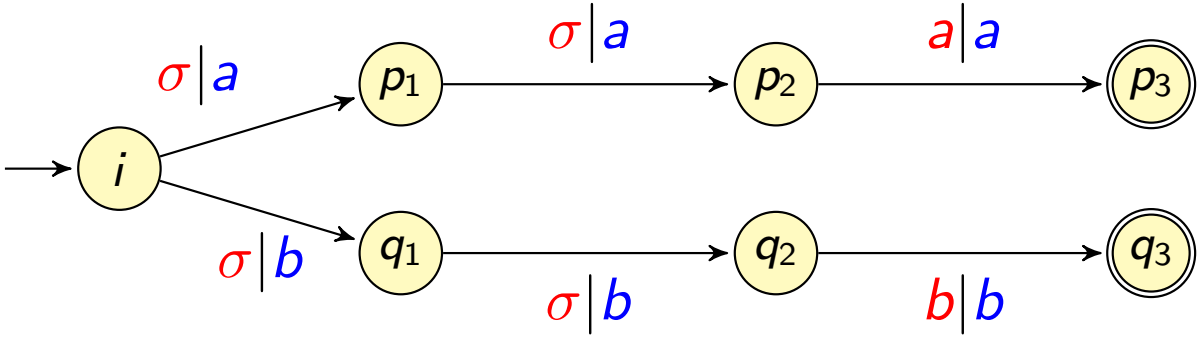

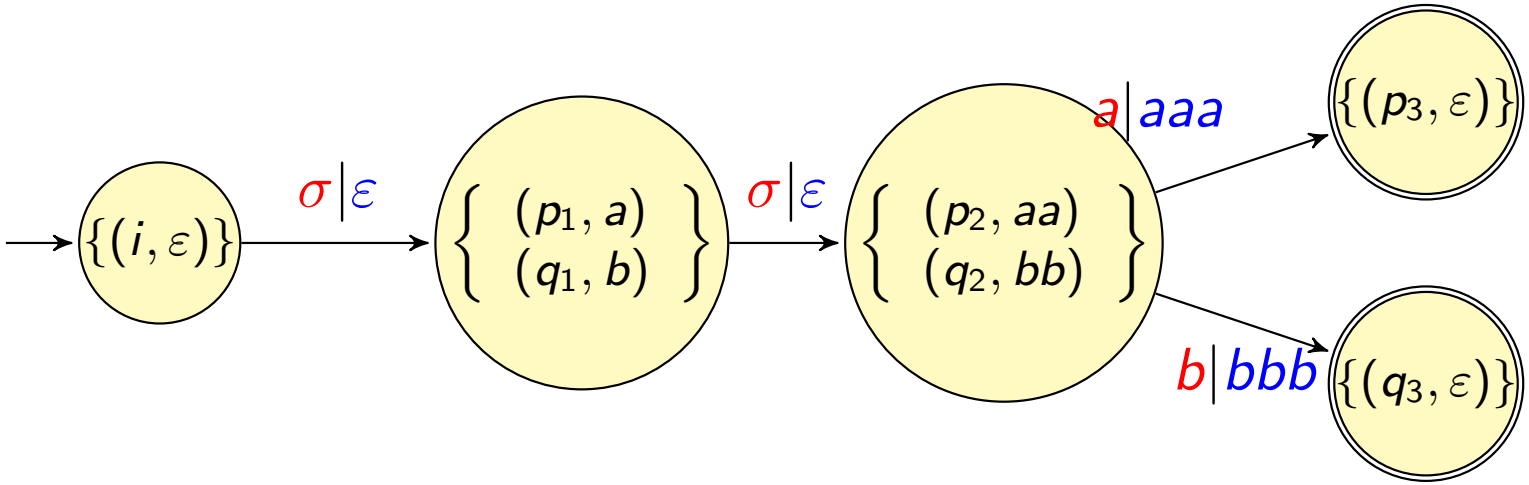

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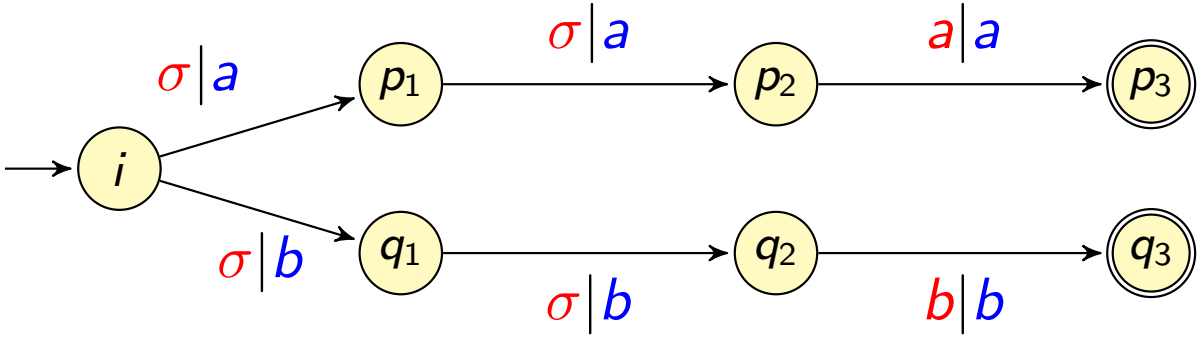
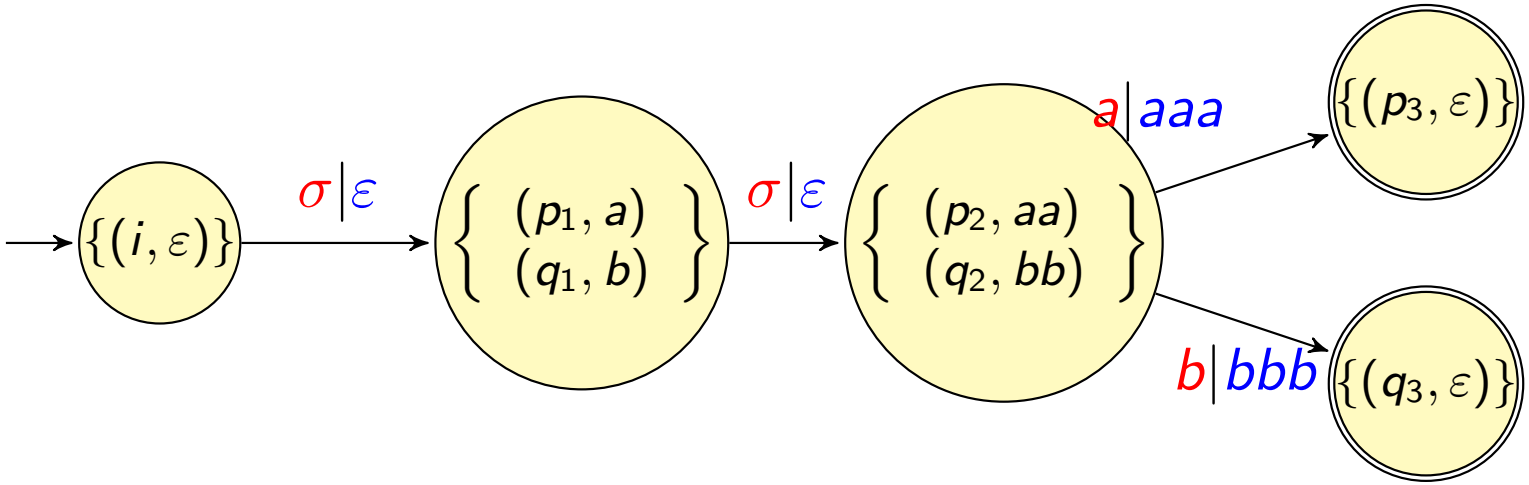




Goal: characterize termination of subset construction

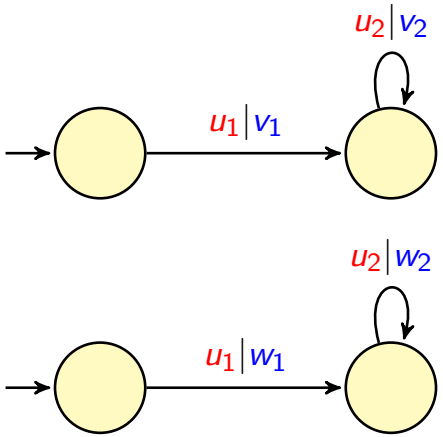
Twinning Property [Choffrut77]

We define:

$$\mathrm{delay}(u, v) = \mathrm{lcp}(u, v)^{-1}.(u, v)$$

Example:
$\mathrm{lcp}(aaa, aab) = aa$
$\mathrm{delay}(aaa, aab) = (a, b)$

For all situations like:



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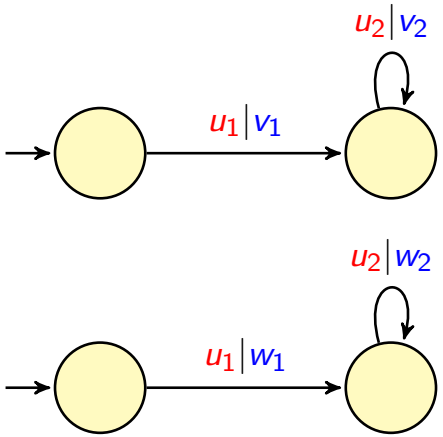

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Lemma

If a fNFT satisfies the Twinning Property, then the delays computed by the subset construction are bounded.

Corollary

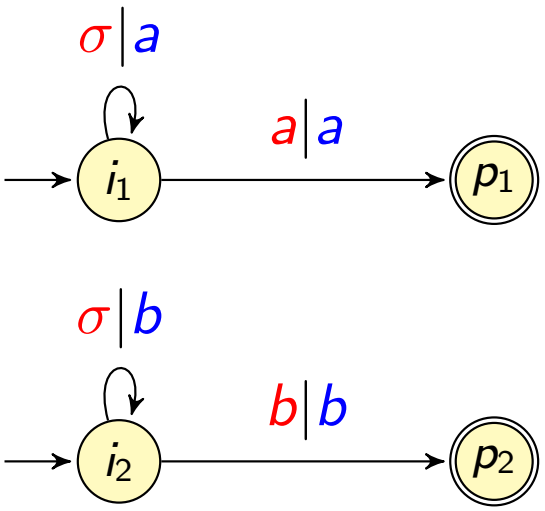
Twinning Property $\iff$ Termination of subset construction.

Theorem ([WK95])

Twinning Property can be decided in PTime.

A counter example

$$dom(f) = \Sigma^*$$
$$f(u) = last(u)^{|u|}$$



After reading an input word u:

- longest common prefix of outputs $= \varepsilon$
- subset construction $= \{(i_1, a^{|u|}),\ (i_2, b^{|u|})\}$

➜ The subset construction does not terminate.

The TP is violated: consider synchronised loops around i_1 and i_2.

Using right-appending streaming string transducers

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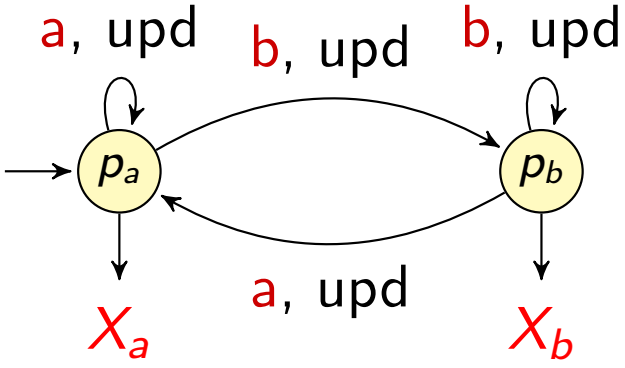

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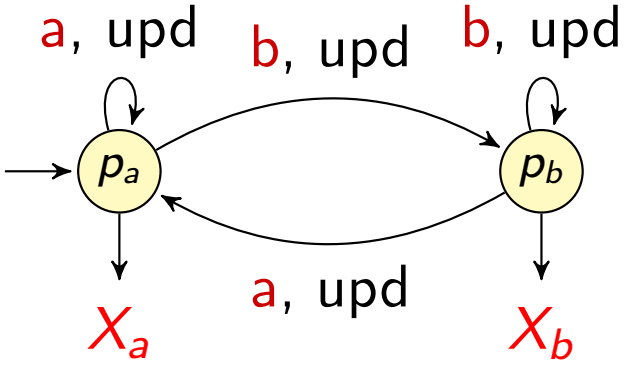

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Can we do better?
No! 1 register is DFT

Register Complexity Problem

Input: A right-appending SST T
Question: Minimal k s.t. there exists a k-raSST T' with $T \equiv T'$

Register complexity using Twinning Property [LICS'16]

Intuition:

2 registers needed if there are 2 runs generating arbitrarily large delays

k registers needed if there are k runs generating pairwise arb. large delays

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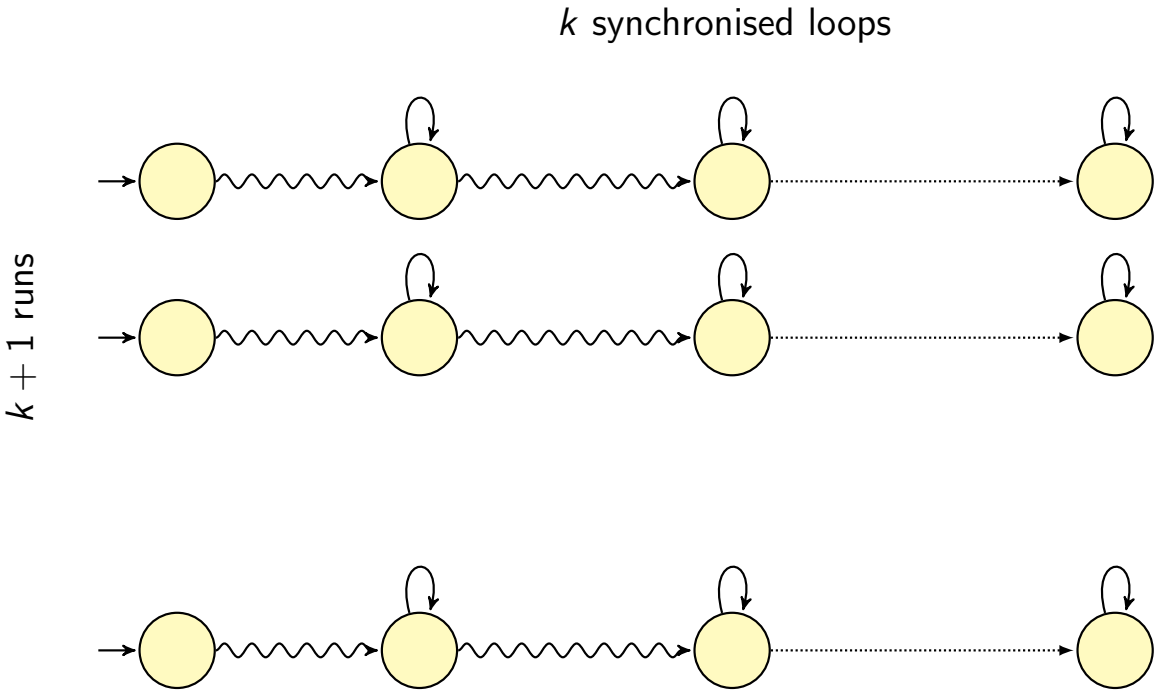
Contraposition: Twinning Property of order k



➜ there are two runs that remain "close"

Register complexity using Twinning Property [LICS'16]

Lemma

If T satisfies the TP of order k, then from any set of runs on the same input word, one can extract k runs such that every run is "close" from one of these k runs.

Theorem

A fNFT is definable by a k-raSST iff it satisfies the TP of order k.

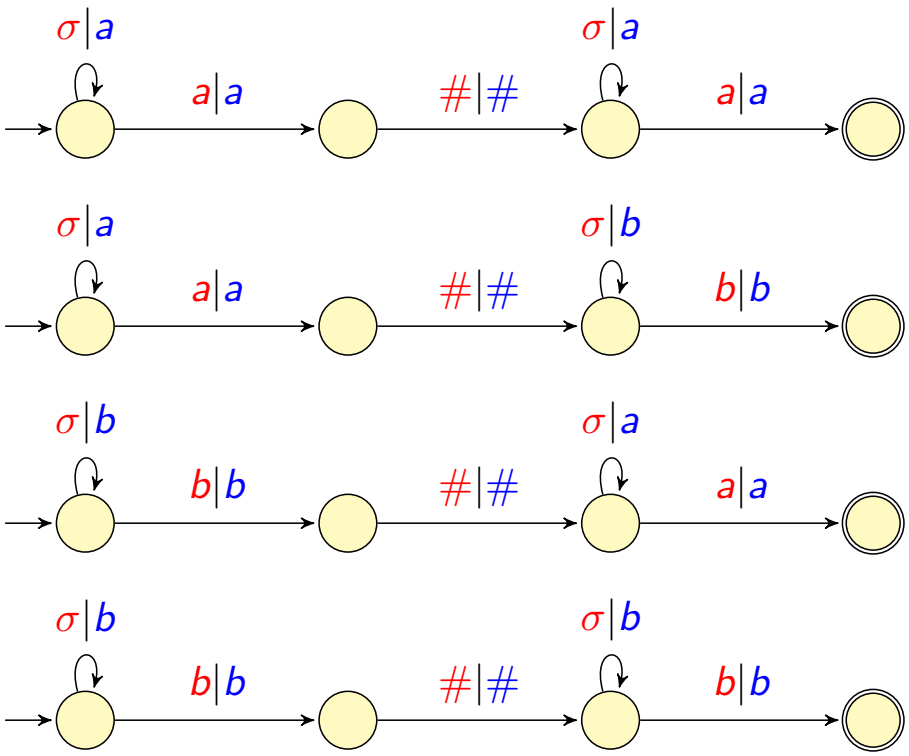
Theorem

Given a fNFT T and k (in unary), deciding whether T satisfies the TP of order k is PSpace-complete.

Register complexity using Twinning Property [LICS'16]

An example: how many registers for the following function?

$$f : u_1\#u_2 \mapsto last(u_1)^{|u_1|}\#last(u_2)^{|u_2|}$$



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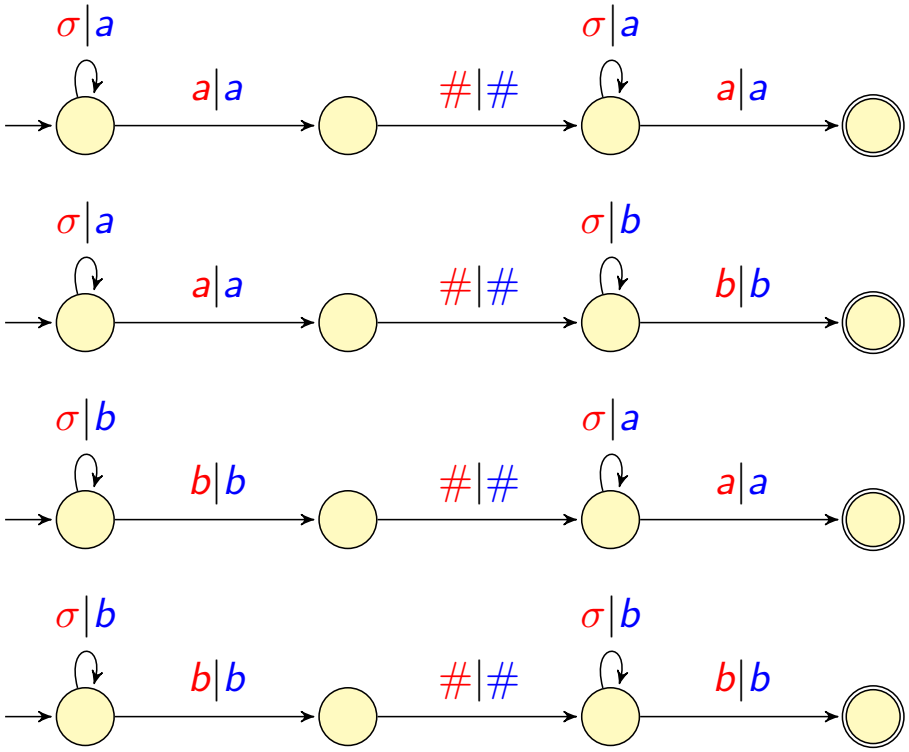

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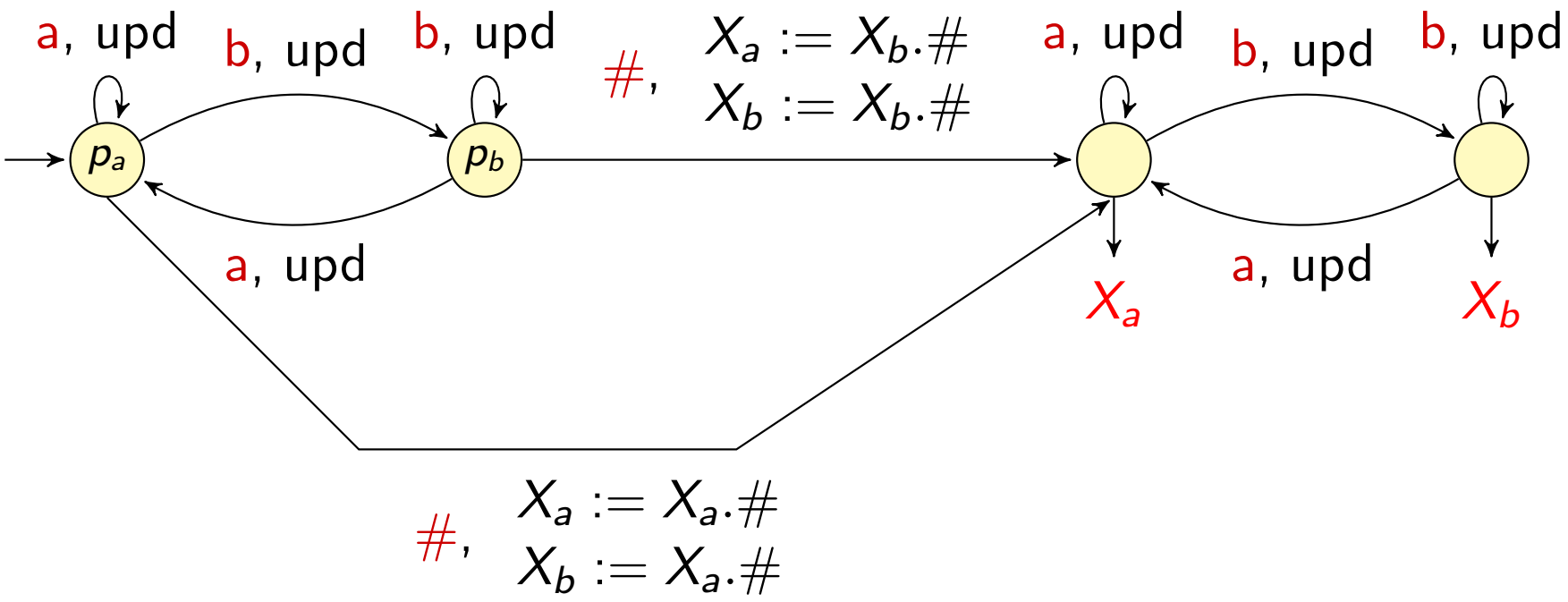

Other patterns: finite valuedness [Weber90]

Definition

A NFT T is finite valued iff $\exists k \mid \forall u \in \Sigma^*, \#[\![T]\!](u) \leq k$.

Theorem

Equivalence of finite valued NFT is decidable.

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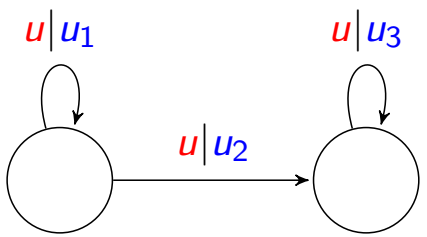
Theorem

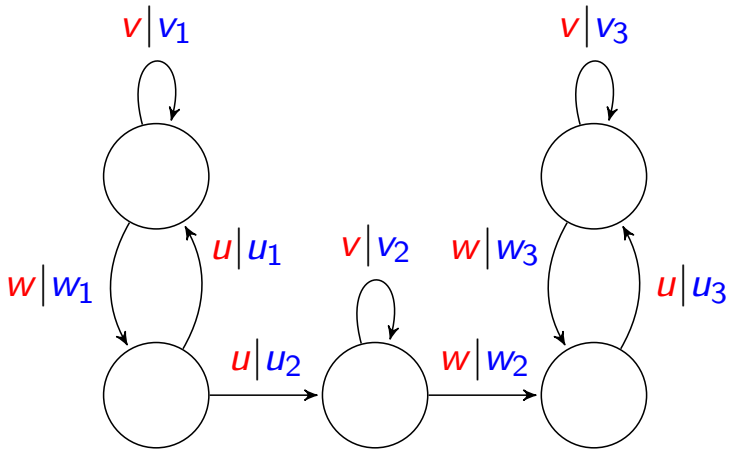
An NFT T is finite-valued iff it satisfies criteria G_0 and G_1.

Theorem

Finite valuedness is decidable in PTime.



we have $u_1 u_2 = u_2 u_3$



we have $|w_1| = |w_2|$

Other patterns: multi-sequential relations [CS86,FJ15]

Definition

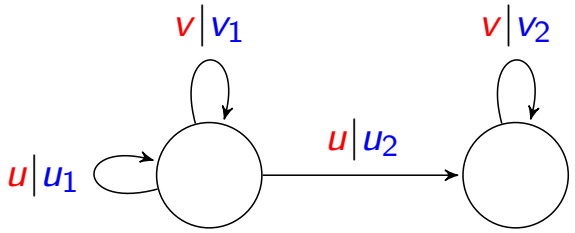
A function/relation R is multi-sequential iff there exists a finite number of DFTs $T_1, \ldots, T_n$ such that $R = \bigcup_i [\![T_i]\!]$

Theorem

An NFT T is multi-sequential iff it satisfies the fork property.

Theorem

Multi sequentiality is decidable in PTime.



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A perspective about multi-sequential functions/relations

Multi-sequential functions $\equiv$ SST with updates $X := X.a$

Register complexity in this class $\equiv$ Minimization of size of the union

➜ Find the right Twinning Property?

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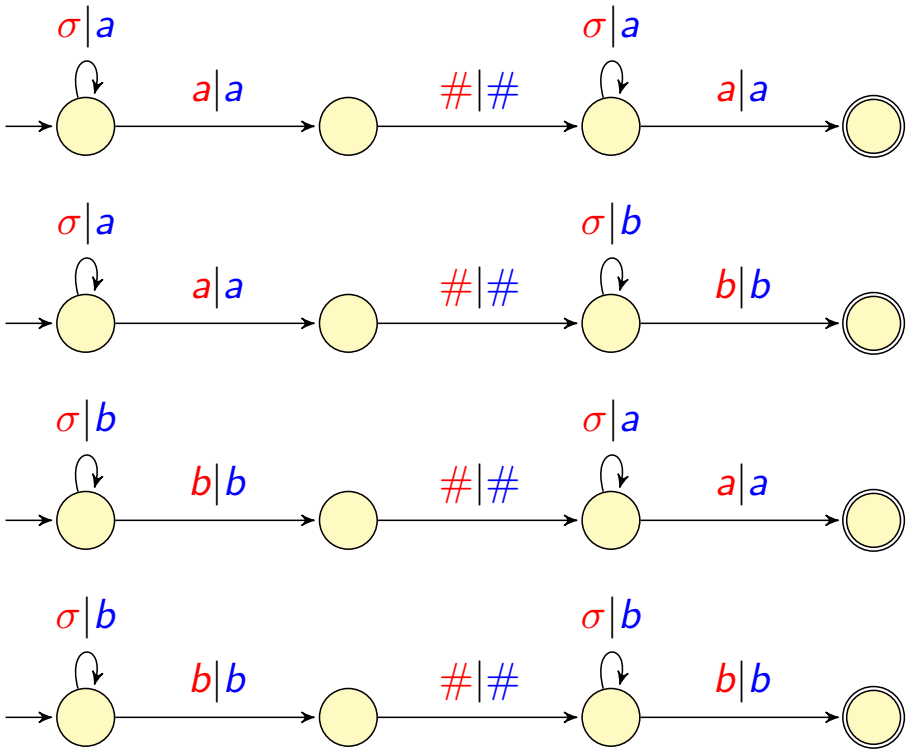
$$f : u_1 \# u_2 \mapsto$$

$$last(u_1)^{|u_1|} \# last(u_2)^{|u_2|}$$

Minimum=4



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$u \sim_L v \iff \forall w, w', wuw' \in L$ iff $wvw' \in L$

Syntactic monoid: $\Sigma^* / \sim_L$

It is isomorphic to the transition monoid of the minimal automaton of L.

MSO Transductions (Courcelle)

- input string seen as the logical structure over $\{succ, (lab_a)_{a \in \Sigma}\}$
- output predicates defined with MSO formulas interpreted over the input structure

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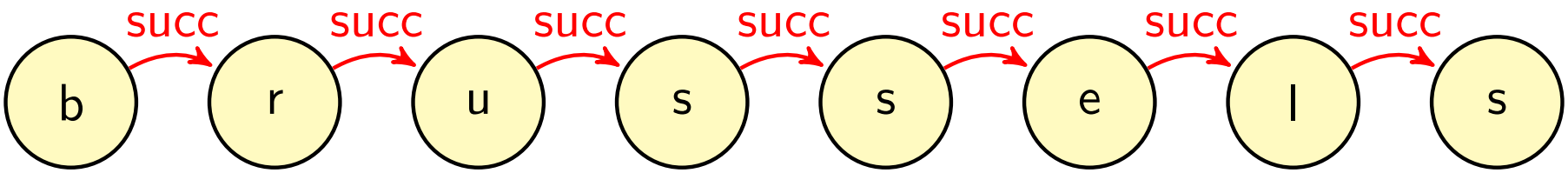

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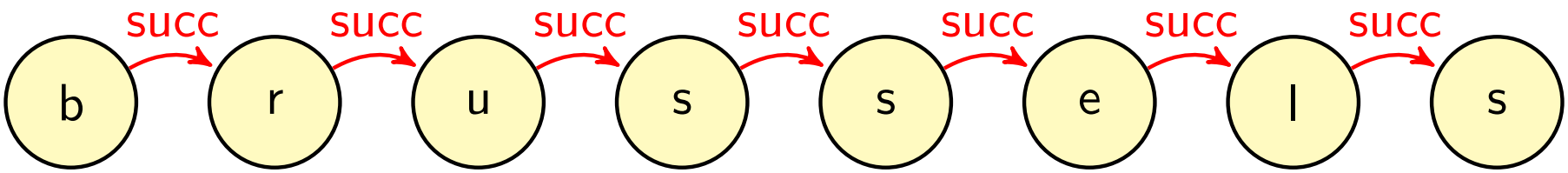

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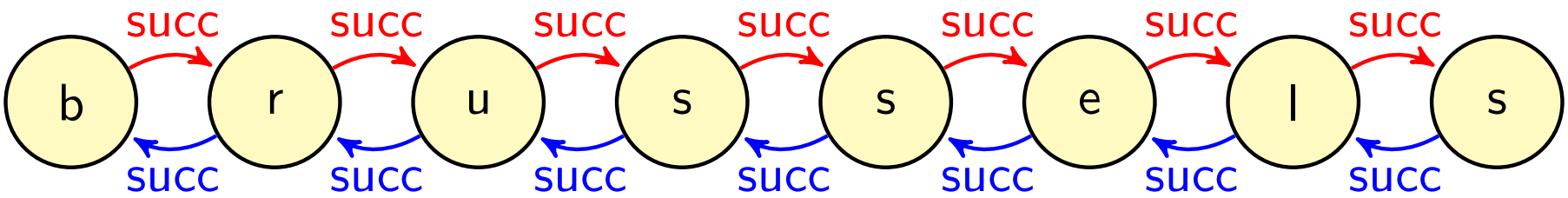

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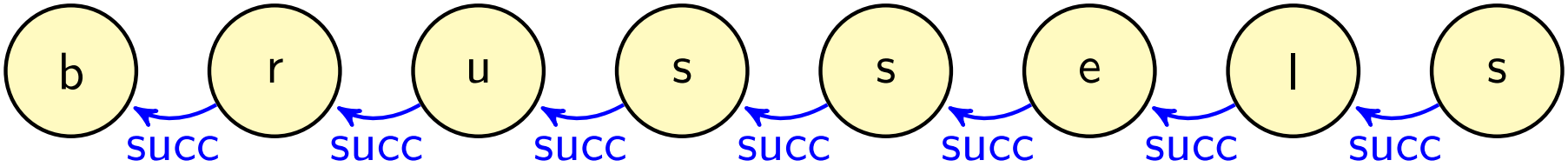

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Equivalence

- Given two MSOT ϕ_1, ϕ_2, does $[\![\phi_1]\!] = [\![\phi_2]\!]$ hold ?
  1. Transform ϕ_1, ϕ_2 into 2DFT T_1, T_2
  2. Check equivalence of T_1 and T_2

Order-preserving MSOT

- no backward edges in the MSO graph.
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- syntactic presentation: guard formula $\phi_{succ}(x, y)$ by $x \leq y$.

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Theorem ([Bojanczyk14,Filiot15])

Rational functions coincide with order-preserving MSO transductions.

As a corollary of 1-Way definability of 2-Way transducers, we have:

Theorem

It is decidable, given an MSOT ϕ, whether ϕ is equivalent to some order-preserving MSOT.

Sequential functions [Choffrut 03]

Key: consider earliest transducers (produce output asap)

What can be produced after reading input u?

$$\hat{f}(u) = \mathrm{lcp}(\{f(uv) \mid uv \in dom(f)\})$$

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$f : \Sigma^* \to \Sigma^*$ is sequential iff $\approx_f$ has finite index.

Classes of $\approx_f$ can be used to define a minimal DFT:

$$[u]_{\approx_f} \xrightarrow{a|v} [ua]_{\approx_f} \text{ where } v = \hat{f}(u)^{-1}.\hat{f}(ua)$$

Rational functions [Reutenauer, Schutzenberger 01]

Function $f : u \to last(u)^{|u|}$ is not sequential

➜ needs some information about the suffix of the input word (look-ahead)

- may be infinite: $f : u_1\#u_2\#u_3 \ldots \mapsto last(u_1)^{|u_1|}\#last(u_2)^{|u_2|}\# \ldots$
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Original left congruence: $u \leftharpoonup_f v$ iff

- $dom(f).u^{-1} = dom(f).v^{-1}$
- Functions $w \mapsto f(wu)$ and $w \mapsto f(wv)$ are "adjacent"

Adjacent: $sup_w |\text{delay}(f(wu), f(wv))| < +\infty$

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Theorem

$f : \Sigma^* \to \Sigma^*$ is rational iff $\leftarrow_f$ has finite index.

In addition, definition of a canonical transducer.

Rational functions: an example

Function $f : u \to last(u)^{|u|}$ with $\Sigma = \{a, b\}$

Classes of $\leftharpoonup_f$: $[\varepsilon] = \{\varepsilon\}$ $\quad [a] = \Sigma^* a$ $\quad [b] = \Sigma^* b$

Indeed, we have $xa \leftharpoonup_f ya$:

$$f(wxa) = a^{|w|+|x|+1} \quad \text{and} \quad f(wya) = a^{|w|+|y|+1}$$

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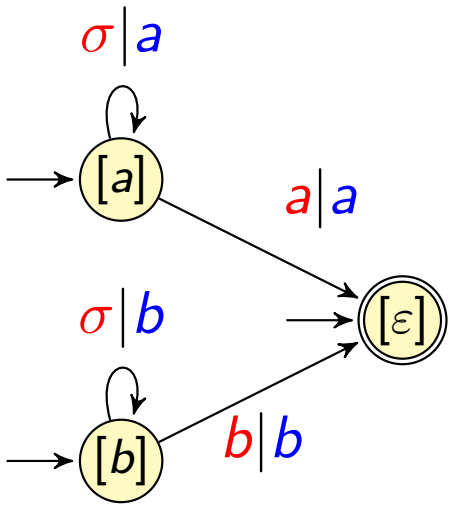
The fNFT guesses at each step the class (for $\leftharpoonup_f$) of the suffix



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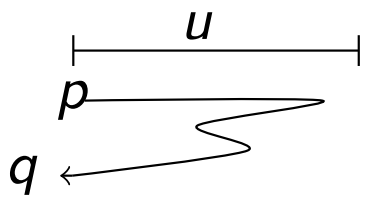

Similarly for left-to-right, right-to-left...

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Define $u \sim_T v \iff \forall x, y \in \{L, R\}, Runs_{xy}(u) = Runs_{xy}(v)$

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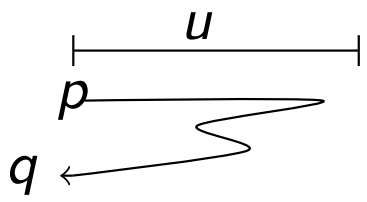

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Congruence of the transition monoid:
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Does not depend on the outputs!

Can also be defined for SST (taking into account the register updates)

First-Order definable transformations

Theorem (First-Order definable languages)

Given a language $L \subseteq \Sigma^*$, the following are equivalent:

- *L is definable in $FO(<, \{p_a, a \in \Sigma\})$*
- *L is accepted by an automaton whose transition monoid is aperiodic*
- *L is star-free*
- *L is definable in LTL (linear temporal logic)*

Aperiodic monoid M: $\exists n \mid \forall m \in M, n^n = m^{n+1}$

First-Order definable transformations

Theorem ([FGL16])

Let T be a fNFT. Then $[\![T]\!]$ is order-pres. FOT definable iff its canonical transducer is aperiodic.

Corollary

Order-preserving FOT is decidable among rational functions.

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Theorem ([FKT14,CD15,DJR16])

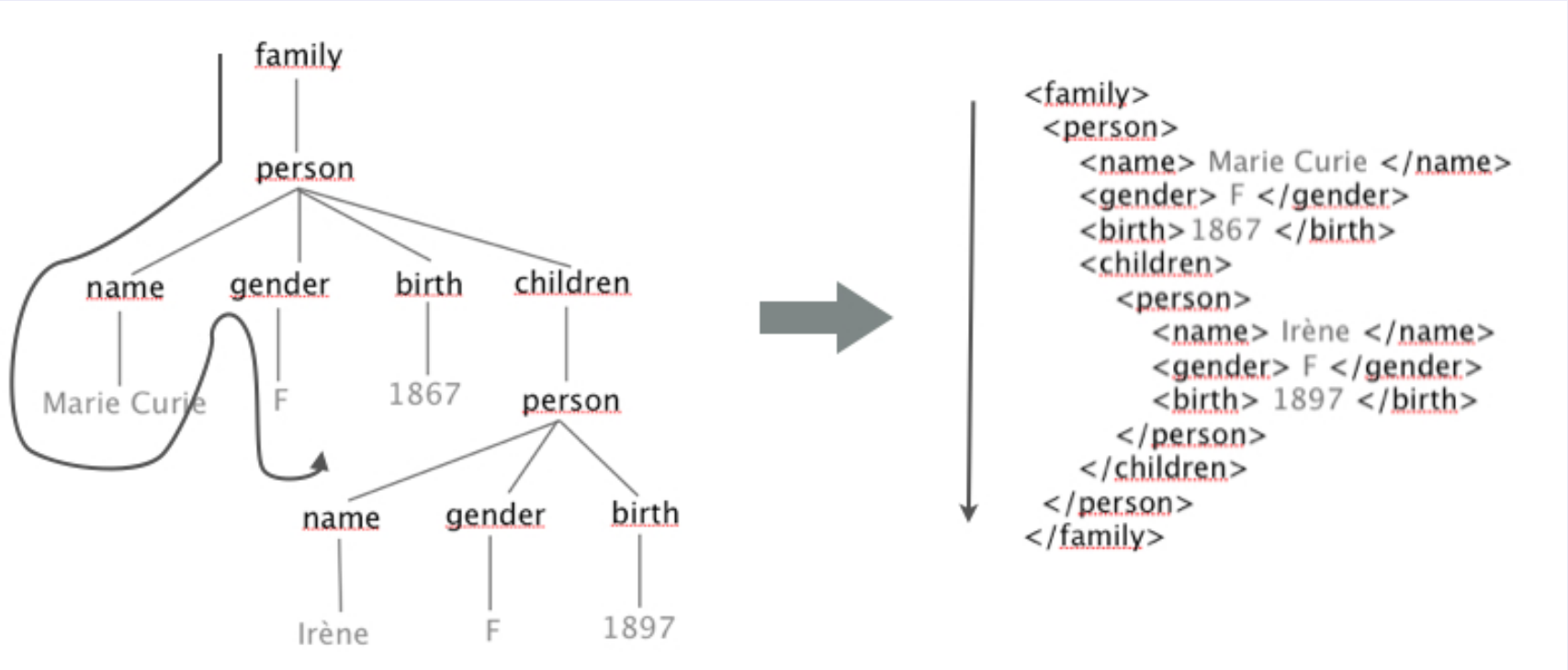
Let $f : \Sigma^* \to \Sigma^*$. The following are equivalent:

- *f is FOT-definable*
- *f is definable by an aperiodic 2DFT*
- *f is definable by an aperiodic copyless SST*

Overview

Motivation: Streaming XML Transformations

XML document ≡ tree linearization



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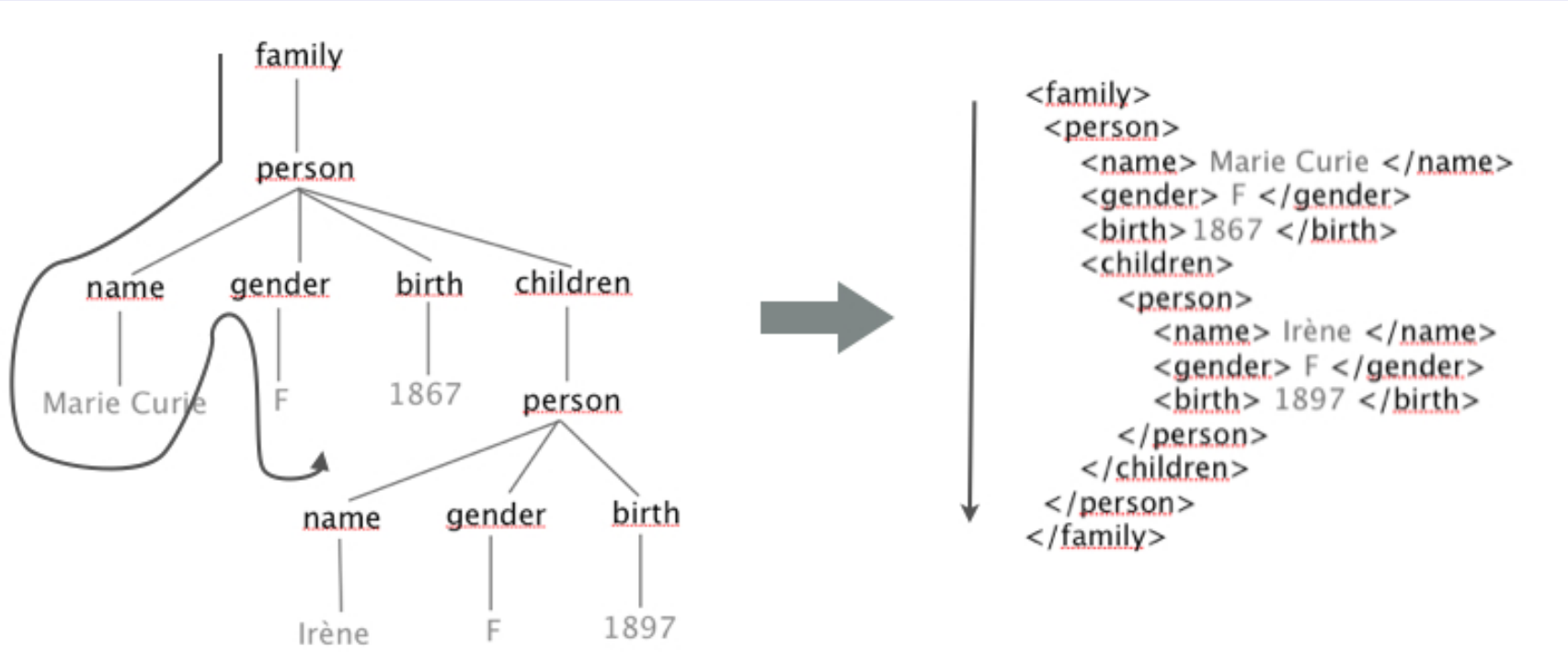

XML documents are words with a nesting structure:

Structured alphabet: call symbols, return symbols, internal symbols

Visibly Pushdown Automata (VPAs) [AM04]

VPAs = Pushdown Automata on *structured* alphabet $\Sigma = \Sigma_c \uplus \Sigma_r \uplus \Sigma_i$:

- push **one** stack symbol on call symbols Σ_c
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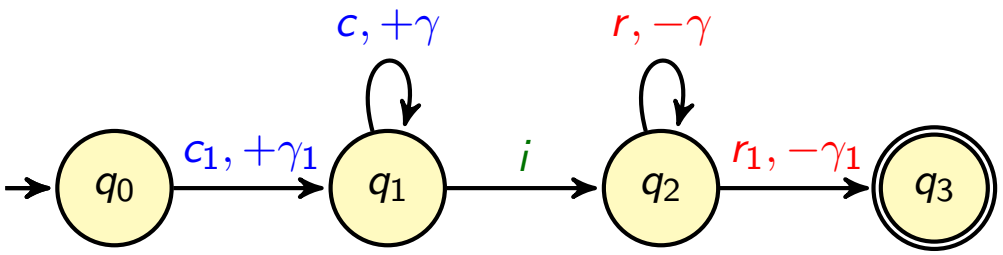
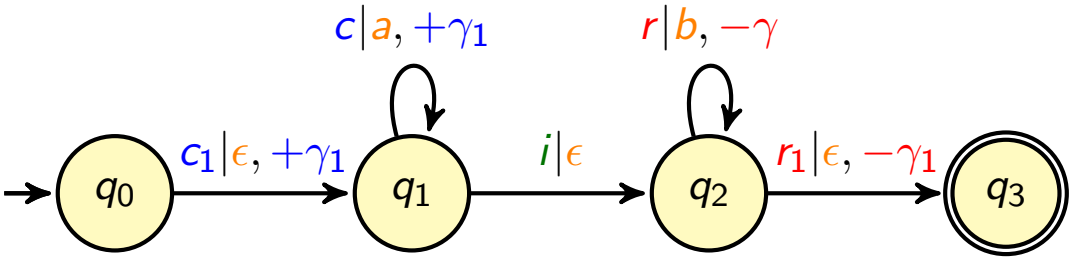


- closure properties
- decidability properties
- determinizable

$$L(A) = \{c_1 c^n \ i \ r^n \ r_1 \ \mid \ n \geq 0\}$$

Visibly Pushdown Transducers (VPTs) [RS08],[MFCS'10]

= associate output words in Δ^* with transitions of a VPA



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- nested word-to-word
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- functional VPT
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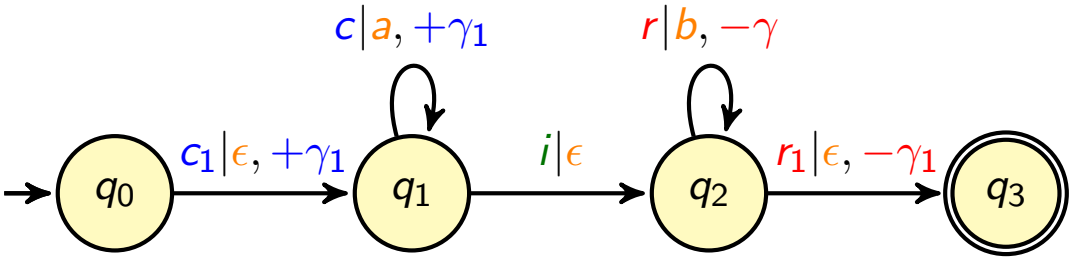

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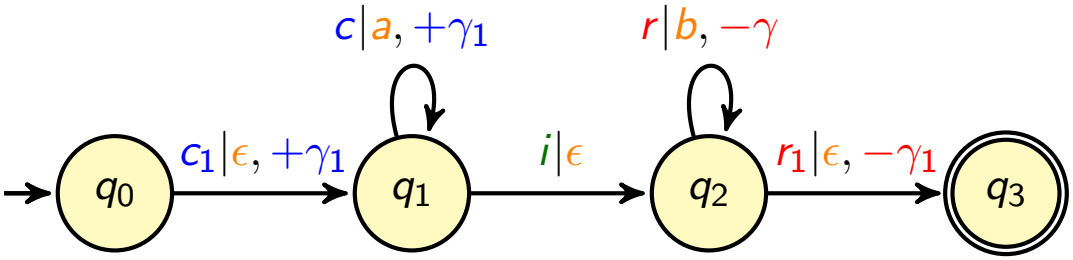

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- Well-nested output words: composition and type-checking [DLT'14]

Streamability Problem for VPT [FSTTCS'11]

Streaming evaluation: avoid the storage of the whole input

Fix a functional VPT T.

How much memory is needed to compute $T(u)$ from an input stream u?

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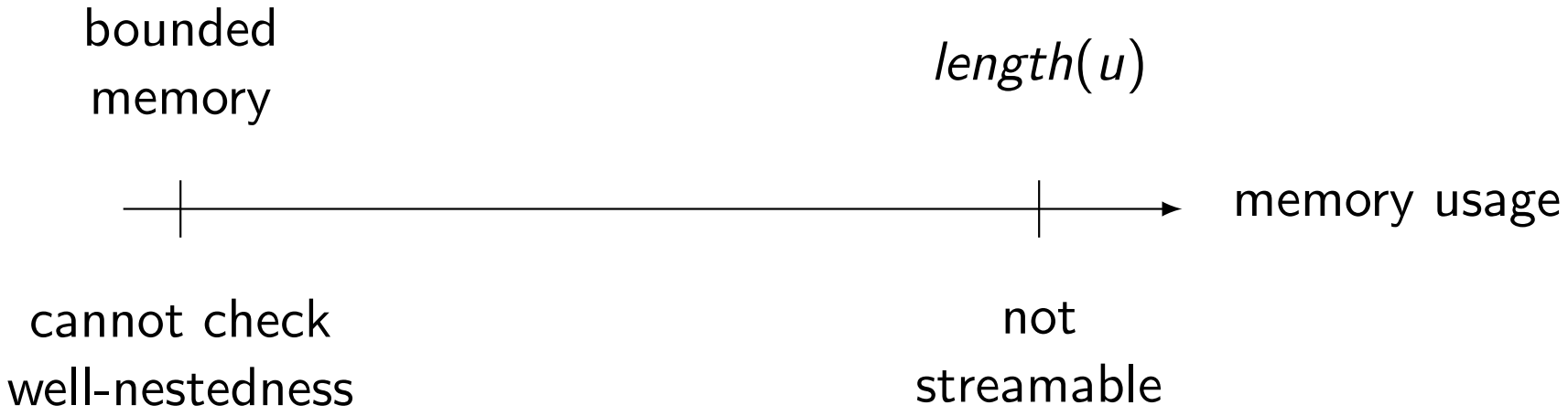

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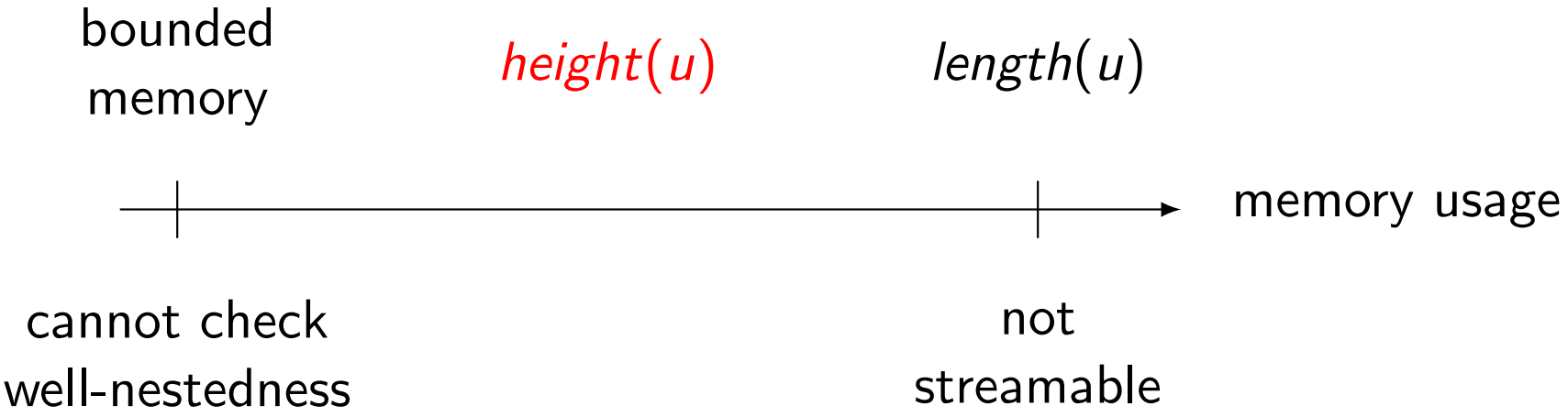

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Input: a transformation T defined by a fVPT

Output: can T be realized with height-bounded memory?

$$\exists f : \mathbb{N} \to \mathbb{N} \cdot \forall u \in dom(T)$$

$T(u)$ can be computed with $f(height(u))$-bounded memory?

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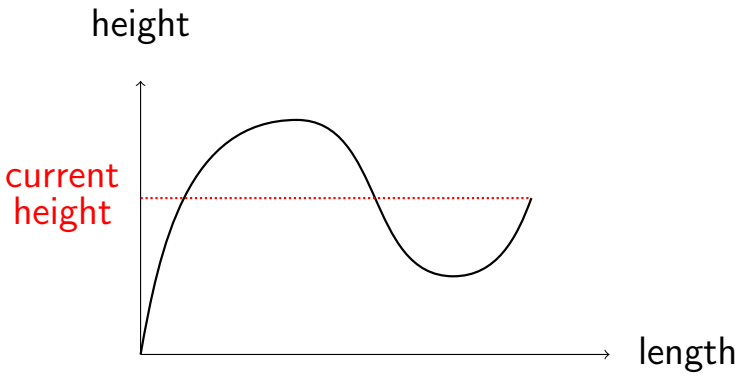
Using a twinning property, we prove:

Decidable in Co-NPTime

But: memory may depend exponentially in $height(u)$

Online Bounded Memory [FSTTCS'11]

Current height:



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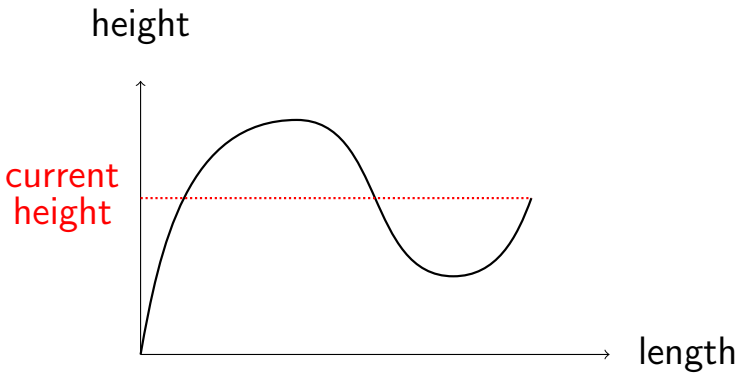
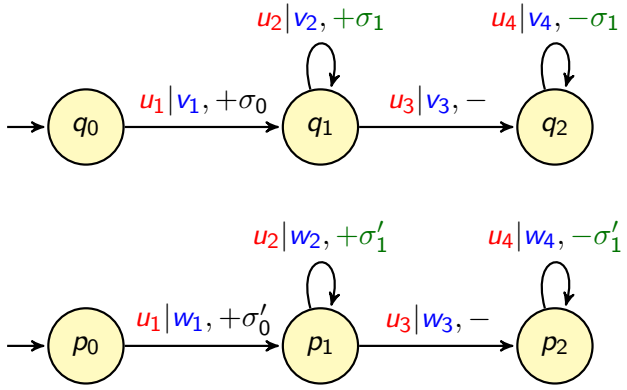


Matched Twinning Property



$\text{delay}(v_1, w_1) = \text{delay}(v_1 v_2 v_3 v_4, w_1 w_2 w_3 w_4)$

Online Bounded Memory Problem

Input: a transformation T defined by a fVPT

Output: can T be realized with current-height-bounded memory?

Characterization : Matched Twinning Property

➜ ensures quadratic dependence in the current height

➜ Decidable in Co-NPTime

Logical characterization for VPTs [LICS'16]

Theorem

MSO definable nested word-to-word transformations are expressively equivalent to 2DVPT$_{su}$.

- Two-way VPTs (stack operation exchanged when reading backwards)
- su : single-use

Key result: 2DVPT^{LA} $\equiv$2DVPT (LA:Look Around)

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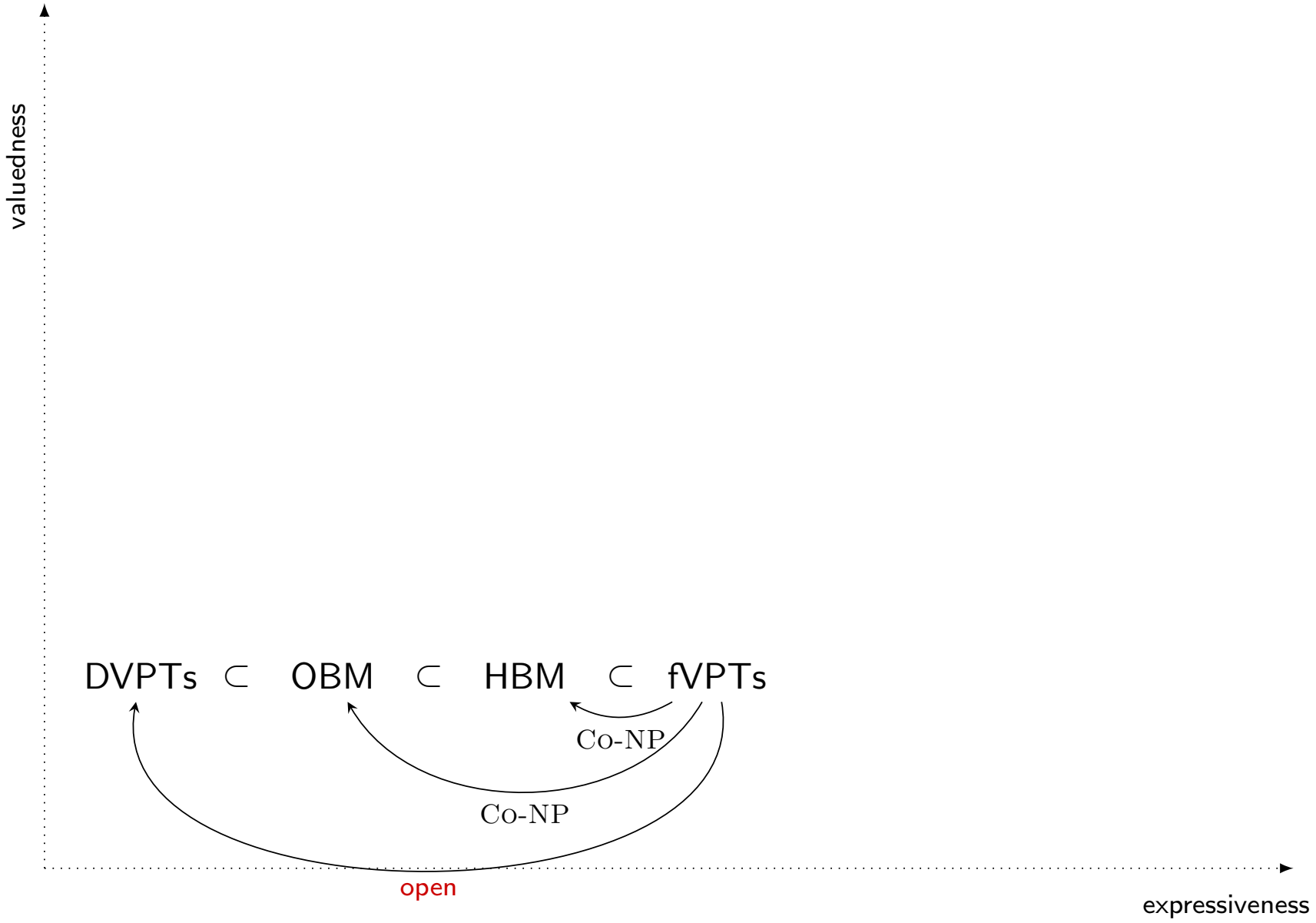
Bonus:

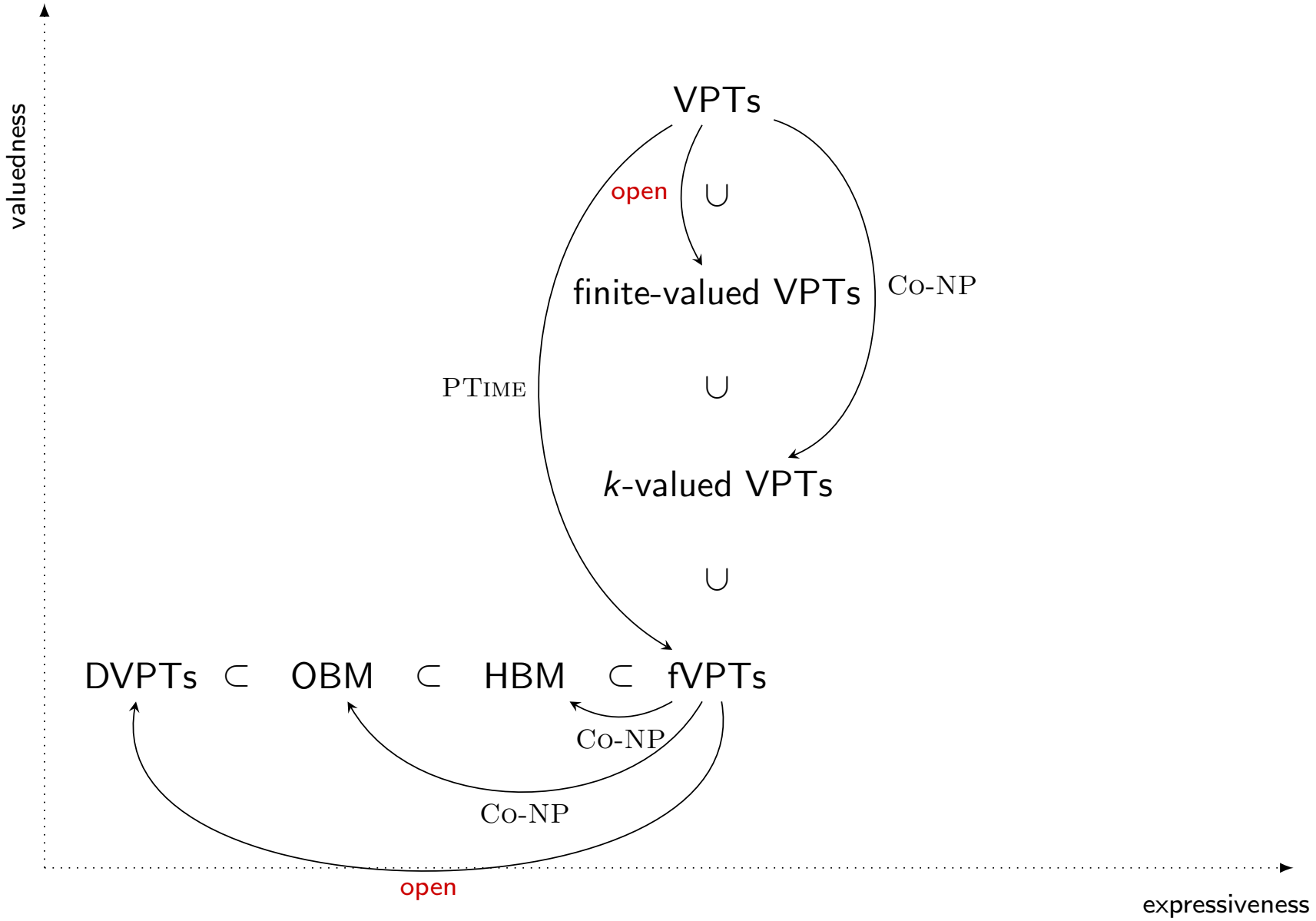
Theorem

Order-preserving MSO definable nested word-to-word transformations are expressively equivalent to functional VPTs.

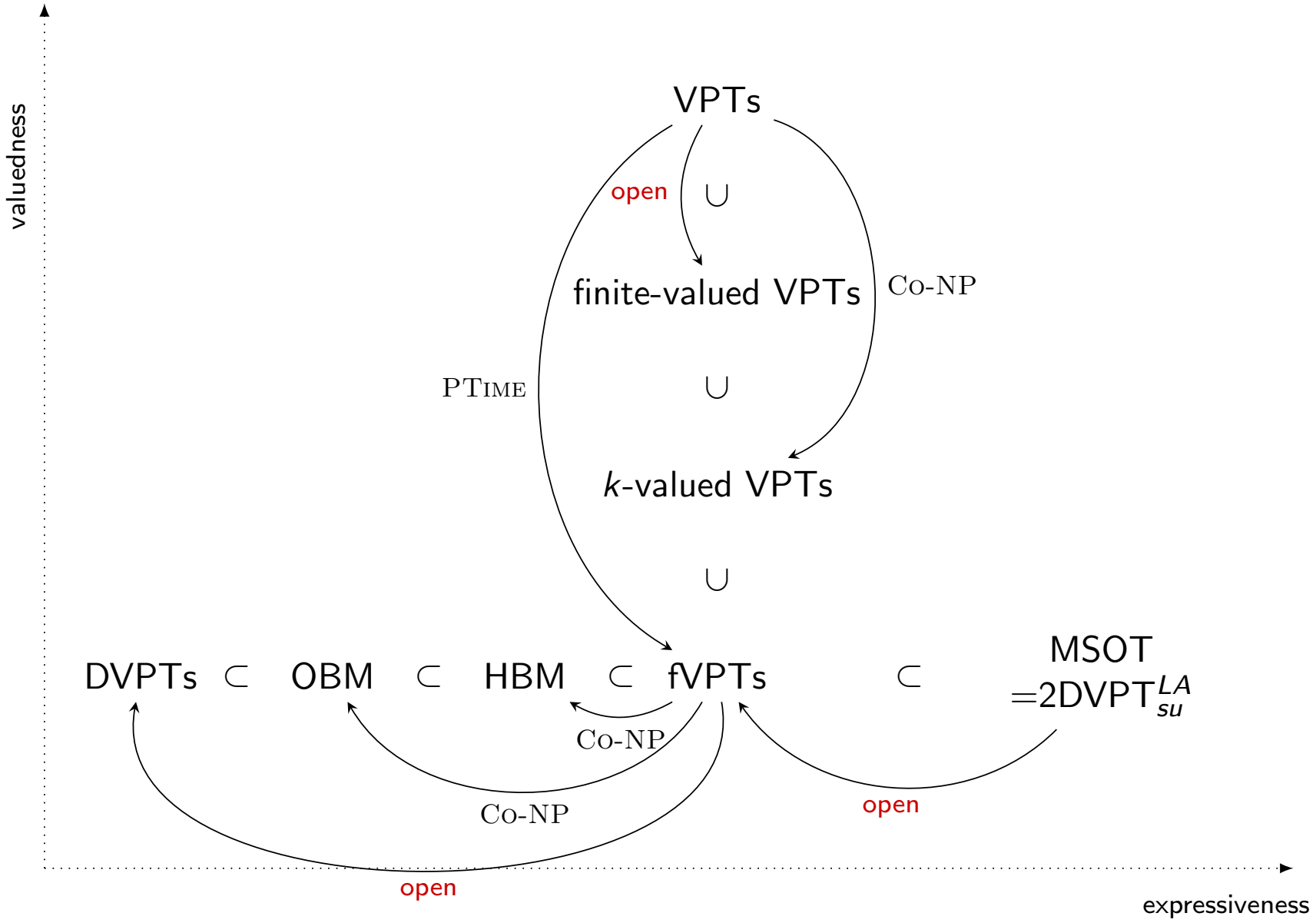
Summary for nested words transductions



Summary for nested words transductions



Summary for nested words transductions



Overview

Summary

- Decidability results based on patterns
  - Shift from NFT to raSST
  - Allows to derive efficient decision procedures
- Logic and Algebra connections
  - Minimization for sequential functions
  - Decidability of FOT for rational functions
  - Characterization only for regular functions
- Nested words
  - Most of the decidability results can be lifted
  - Logic can be lifted too
  - Algebra is missing

I did not present...

Some of these results are valid not only for transducers.

Recent results on model simplification:

- minimize the number of passes of a 2DFT
- minimize the number of registers of non-det. SST

Important results about relations:

- k-valuedness is decidable
- k-valued relations can be decomposed into functions

Recent trend about an alternative semantics (see next talk)

Perspectives

Shift from rational to regular functions

- register minimization
- algebraic presentation, canonical object

Specification languages for transformations

Alternative semantics to break undecidability/high complexity

Perspectives

Shift from rational to regular functions

- register minimization
- algebraic presentation, canonical object

Specification languages for transformations

Alternative semantics to break undecidability/high complexity

Thanks!