Shortest path problem in rectangular complexes of global nonpositive curvature

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Abstract. CAT(0) metric spaces constitute a far-reaching common generalization of Euclidean and hyperbolic spaces and simple polygons: any two points x and y of a CAT(0) metric space are connected by a unique shortest path $\gamma(x,y)$. In this paper, we present an efficient algorithm for answering two-point distance queries in CAT(0) rectangular complexes and two of theirs subclasses, ramified rectilinear polygons (CAT(0) rectangular complexes in which the links of all vertices are bipartite graphs) and squaregraphs (CAT(0) rectangular complexes arising from plane quadrangulations in which all inner vertices have degrees ≥ 4). Namely, we show that for a CAT(0) rectangular complex $\mathcal{K}$ with n vertices, one can construct a data structure $\mathcal{D}$ of size $O(n^2)$ so that, given any two points $x,y \in \mathcal{K}$, the shortest path $\gamma(x,y)$ between x and y can be computed in $O(d(p,q))$ time, where p and q are vertices of two faces of $\mathcal{K}$ containing the points x and y, respectively, such that $\gamma(x,y) \subset \mathcal{K}(I(p,q))$ and $d(p,q)$ is the distance between p and q in the underlying graph of $\mathcal{K}$. If $\mathcal{K}$ is a ramified rectilinear polygon, then one can construct a data structure $\mathcal{D}$ of optimal size $O(n)$ and answer two-point shortest path queries in $O(d(p,q)\log\Delta)$ time, where Δ is the maximal degree of a vertex of $G(\mathcal{K})$. Finally, if $\mathcal{K}$ is a squaregraph, then one can construct a data structure $\mathcal{D}$ of size $O(n\log n)$ and answer two-point shortest path queries in $O(d(p,q))$ time.

Keywords. Shortest path problem, rectangular complex, geodesic l_2-distance, global nonpositive curvature.

1 Introduction

The shortest path problem is one of the best-known algorithmic problems with many applications in routing, robotics, operations research, motion planning, urban transportation, and terrain navigation. This fundamental problem was intensively studied both in discrete settings like graphs and networks (see, e.g., Ahuja, Magnanti, and Orlin [1]) as well as in geometric spaces (simple polygons, polygonal domains with obstacles, polyhedral surfaces, terrains; see, e.g., Mitchell [36]). In the case of graphs $G=(V,E)$ in which all edges have non-negative lengths, a well-known algorithm of Dijkstra allows to compute a tree of shortest paths from any source vertex to all other vertices of the graph. In simple polygons P

endowed with the (intrinsic) geodesic distance, each pair of points $p, q \in P$ *can be connected by a unique shortest path.* Several algorithms for computing shortest paths inside a simple polygon are known in the literature [27, 28, 31, 34, 41], and all are based on a triangulation of P in a preprocessing step (which can be done in linear time due to Chazelle's algorithm [12]). The algorithm of Lee and Preparata [34] finds the shortest path between two points of a triangulated simple polygon in linear time (*two-point shortest path queries*). Given a source point, the algorithm of Reif and Storer [41] produces in $O(n \log n)$ time a search structure (in the form of a shortest path tree) so that the shortest path from any query point to the source can be found in time linear in the number of edges of this path (the so-called *single-source shortest path queries*). Guibas et al. [28] return a similar search structure, however their preprocessing step takes only linear time once the polygon is triangulated (see Hersberger and Snoeyink [30] for a significant simplification of the original algorithm of [28]). Finally, Guibas and Hersberger [27] showed how to preprocess a triangulated simple polygon P in linear time to support shortest-path queries between any two points $p, q \in P$ in time proportional to the number of edges of the shortest path between p and q. Note that the last three mentioned algorithms also return in $O(\log n)$ time the distance between the queried points. In case of shortest path queries in general polygonal domains D with holes, the simplest approach is to compute at the preprocessing step the visibility graph of D. Now, given two query points p, q, to find a shortest path between p and q in D (this path is no longer unique), it suffices to compute this path in the visibility graph of D augmented with two vertices p and q and all edges corresponding to vertices of D visible from p or q; for a detailed description of how to efficiently construct the visibility graph, see the survey [36] and the book [20]. An alternative paradigm is the so-called *continuous Dijkstra* method, which was first applied to the shortest path problem in general polygonal domains by Mitchell [35] and subsequently improved to a nearly optimal algorithm by Hershberger and Suri [31]; for an extensive overview of this method and related references, see again the survey by Mitchell [36].

In this paper, we present an algorithm for efficiently solving two-point shortest path queries in CAT(0) rectangular complexes, i.e., rectangular complexes of global non-positive curvature. CAT(0) metric spaces have been introduced by M. Gromov in his seminal paper [26] and investigated in many recent mathematical papers; in particular, CAT(0) spaces play a vital role in geometric group theory CAT(0) metric spaces can be characterized as the geodesic metric spaces in which any two points can be joined by a unique geodesic shortest path, therefore they represent a far-reaching generalization of geodesic metrics in simple polygons. Several papers are devoted to algorithmic problems in particular CAT(0) spaces. For example, the recent paper by Fletcher et al. [25] investigates algorithmic questions related to computing approximate convex hulls and centerpoints of point-sets in the CAT(0) metric space $P(n)$ of all positive definite $n \times n$ matrices. Billera et al. [8] showed that the space of all phylogenetic trees defined on the same set of leaves can be viewed as a CAT(0) cubical complex. Subsequently, the question of whether the distance and the shortest path between two trees in this CAT(0) space can be computed in polynomial (in the number of leaves) time was raised. Recently, Owen and Provan [40] solved this question in the affirmative; the paper [13] reports on the implementation of the algorithm of [40].

The remaining part of the paper is organized in the following way. In the next preliminary section, we introduce CAT(0) metric spaces, CAT(0) box and rectangular complexes, ramified rectilinear polygons, and squaregraphs. We also introduce the two-point shortest path query problem. In Section 3, we show that the shortest path $\gamma(x, y)$ between two points x, y of a CAT(0) box complex $\mathcal{K}$ is contained in the subcomplex induced by the graph interval $I(p, q)$ between two vertices p, q belonging to the cells containing x and y, respectively. Moreover, we show that this subcomplex $\mathcal{K}(I(p, q))$ can be unfolded in the k-dimensional space $\mathbb{R}^k$ (where k is the dimension of a largest cell of $\mathcal{K}(I(p, q))$) in a such a way that the shortest path between any two points is the same in $\mathcal{K}(I(p, q))$ and in the unfolding of $\mathcal{K}(I(p, q))$. In Section 4, we present the detailed description of the algorithm for answering two-point shortest path queries in CAT(0) rectangular complexes and of the data structure $\mathcal{D}$ used in this algorithm. First we show how to compute the unfolding of $\mathcal{K}(I(p, q))$ in $\mathbb{R}^2$ efficiently. Then we describe the data structure $\mathcal{D}$ and show how to use it to compute the boundary paths of $\mathcal{K}(I(p, q))$. $\mathcal{D}$ is different for general CAT(0) rectangular complexes, for ramified rectilinear polygons, and for squaregraphs. We conclude with a formal description of the algorithm and the analysis of its complexity.

2 Preliminaries

2.1 CAT(0) metric spaces

Let (X, d) be a metric space. A *geodesic* joining two points x and y from X is the image of a (continuous) map γ from a line segment $[0, l] \subset \mathbb{R}$ to X such that $\gamma(0) = x, \gamma(l) = y$ and $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in [0, l]$. The space (X, d) is said to be *geodesic* if every pair of points $x, y \in X$ is joined by a geodesic [11]. A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three distinct points in X (the vertices of Δ) and a geodesic between each pair of vertices (the sides of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x'_1, x'_2, x'_3)$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x'_i, x'_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A geodesic metric space (X, d) is defined to be a *CAT*(0) *space* [26] if all geodesic triangles $\Delta(x_1, x_2, x_3)$ of X satisfy the comparison axiom of Cartan–Alexandrov–Toponogov:

If y is a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2 and y' is the unique point on the line segment $[x'_1, x'_2]$ of the comparison triangle $\Delta(x'_1, x'_2, x'_3)$ such that $d_{\mathbb{E}^2}(x'_i, y') = d(x_i, y)$ for $i = 1, 2,$ then $d(x_3, y) \le d_{\mathbb{E}^2}(x'_3, y')$.

This simple axiom turns out to be very powerful, because CAT(0) spaces can be characterized in several natural ways (for a full account of this theory consult the book [11]). In particular, a geodesic metric space (X, d) is CAT(0) if and only if any two points of this space can be joined by a unique geodesic. CAT(0) is also equivalent to convexity of the function $f : [0, 1] \to X$ given by $f(t) = d(\alpha(t), \beta(t))$, for any geodesics α and β (which is further equivalent to convexity of the neighborhoods of convex sets). This implies that CAT(0) spaces are contractible.

2.2 CAT(0) rectangular and box complexes

A *rectangular complex* $\mathcal{K}$ is a 2-dimensional cell complex $\mathcal{K}$ whose 2-cells are isometric to axis-parallel rectangles of the l_1-plane. If all 1-cells of $\mathcal{K}$ have equal length, then we call $\mathcal{K}$ a *square complex*; in this case we may assume without loss of generality that the squares of the complex are all unit squares. Square complexes are the 2-dimensional instances of *cubical complexes,* viz. the cell complexes (where cells have finite dimension) in which every cell of dimension k is isometric to the unit cube of $\mathbb{R}^k$. Analogously, the complexes in which all cells are axis-parallel boxes are high-dimensional generalizations of rectangular complexes (we will call them *box complexes*). All complexes occurring in our paper are finite, i.e., they have only finitely many cells.

The 0-dimensional faces of a rectangular or box complex $\mathcal{K}$ are called its *vertices*, forming the vertex set $V(\mathcal{K})$ of $\mathcal{K}$. The 1-dimensional faces of $\mathcal{K}$ are called the *edges* of $\mathcal{K}$, and denoted by $E(\mathcal{K})$. The *underlying graph* of $\mathcal{K}$ is the graph $G(\mathcal{K}) = (V(\mathcal{K}), E(\mathcal{K}))$. Conversely, from any graph G one can derive a cube (or a box complex) by replacing all subgraphs of G isomorphic to cubes of any dimensions by solid cubes (or axis-parallel boxes). We denote any complex obtained in this way by $||G||$ and call it the *geometric realization* of G. A cell complex $\mathcal{K}$ is called *simply connected* if it is connected and every continuous mapping of the 1-dimensional sphere $\mathbb{S}^1$ into $\mathcal{K}$ can be extended to a continuous mapping of the disk $\mathbb{D}^2$ with boundary $\mathbb{S}^1$ into $\mathcal{K}$. The *link* of a vertex x in $\mathcal{K}$ is the graph $\mathrm{Link}(x)$ whose vertices are the 1-cells containing x and where two 1-cells are adjacent if and only if they are contained in a common 2-cell (see [11] for the notion of link in general polyhedral complexes). Given a subset S of vertices of $\mathcal{K}$, we will denote by $\mathcal{K}(S)$ the subcomplex of $\mathcal{K}$ induced by S.

Computationally, a rectangular complex $\mathcal{K}$ is defined in the following way. Each rectangular face R of $\mathcal{K}$ is given by the circular list of four vertices and edges incident to R. For each vertex v of $\mathcal{K}$, the neighborhood of v is given as the link graph $\mathrm{Link}(v)$; for each edge of $\mathrm{Link}(v)$ there is a pointer to the unique rectangular face containing v and the edges of $\mathcal{K}$ incident to v which define this edge of $\mathrm{Link}(v)$. Finally, each point x of $\mathcal{K}$ is given by its (local) coordinates in a rectangular cell $R(x)$ of $\mathcal{K}$ containing x (notice that $R(x)$ is unique if x belongs to the interior of $R(x)$, otherwise x may belong to several rectangular faces).

A rectangular or box complex $\mathcal{K}$ can be endowed with several intrinsic metrics [11] transforming $\mathcal{K}$ into a complete geodesic space. Suppose that inside every cell of $\mathcal{K}$ the distance is measured according to an l_1- or l_2-metric. Then the *intrinsic* l_1- or l_2-*metric* of $\mathcal{K}$ is defined by assuming that the distance between two points $x, y \in \mathcal{K}$ equals the infimum of the lengths of the paths joining them. Here a *path* in $\mathcal{K}$ from x to y is a sequence P of points $x = x_0, x_1 \ldots x_{m-1}, x_m = y$ such that for each $i = 0, \ldots, m-1$ there exists a cell R_i of $\mathcal{K}$ containing x_i and x_{i+1}; the *length* of P is $l(P) = \sum_{i=0}^{m-1} d(x_i, x_{i+1})$, where $d(x_i, x_{i+1})$ is computed inside R_i according to the respective metric. We denote the resulting l_1- and l_2-metrics on $\mathcal{K}$ by d_1 and d_2, respectively.

The *interval* between two points x, y of a metric space (X, d) is the set $I(x, y) = \{z \in X : d(x, y) = d(x, z) + d(z, y)\}$; for example, in Euclidean spaces, the interval $I(x, y)$ is the closed line segment having x and y as its endpoints. A subspace Y of a metric space (X, d) is *gated* if for every point $x \in X$ there exists a (unique) point $x' \in Y$, the *gate* of x in Y, such that $d(x,y) = d(x,x') + d(x',y)$ for all $y \in Y$. Gated subspaces are necessarily convex, where a subspace Y of (X,d) is called *convex* if $I(x,y) \subseteq Y$ for any $x,y \in Y$. A *half-space* H of X is a convex subspace with a convex complement. For three points x,y,z of a metric space (X,d), let $m(x,y,z) = I(x,y) \cap I(y,z) \cap I(z,x)$. If $m(x,y,z)$ is a singleton for all $x,y,z \in X$, then the space X is called *median* [3, 43] and we usually refer to $m(x,y,z)$ as to the *median* of x,y,z (here we do not distinguish between the singleton and the corresponding point). A graph G is a *median graph* if (V, d_G) is a median space, where d_G is the standard graph-metric of G. Discrete median spaces, in general, can be regarded as median networks: a *median network* is a median graph with weighted edges such that opposite edges in any 4-cycle have the same length [2]. Median graphs not containing any induced cube (or cube network, respectively) are called *cube-free.* A *median complex* is a cube or a box complex of a median graph. We will say that a subset S of points of a CAT(0) rectangular complex $\mathcal{K}$ is d_i*-convex* for $i = 1, 2$, if S is a convex subset of the metric space $(\mathcal{K}, d_i)$ (convex subsets of the underlying graph $G(\mathcal{K})$ will be called *graph-convex*).

Now we recall the combinatorial characterization of CAT(0) cubical and box complexes given by Gromov.

Theorem 2.1 *[26] A cubical (or a box) polyhedral complex $\mathcal{K}$ with the intrinsic l_2-metric d_2 is CAT(0) if and only if $\mathcal{K}$ is simply connected and satisfies the following condition: whenever three $(k+2)$-cubes of $\mathcal{K}$ share a common k-cube and pairwise share common $(k+1)$-cubes, they are contained in a $(k+3)$–cube of $\mathcal{K}$.*

In some recent papers, CAT(0) cubical polyhedral complexes were called cubings. With some abuse of language, we will call *cubings* all CAT(0) box complexes. The following relationship holds between cubings and median polyhedral complexes (this result was used in several recent papers in geometric group theory [14, 39]).

Theorem 2.2 *[15, 42] Median complexes and cubings (both equipped with the l_2-metric) constitute the same objects.*

In this paper we will mainly investigate the CAT(0) rectangular complexes (i.e., 2-dimensional cubings), which can be characterized in the following way:

Theorem 2.3 *[5] For a rectangular complex $\mathcal{K}$ the following conditions are equivalent:*

(i) the underlying graph $G(\mathcal{K})$ of $\mathcal{K}$ is a cube-free median graph;

(ii) the metric space $(\mathcal{K}, d)$ is median;

(iii) $\mathcal{K}$ equipped with the intrinsic l_2-metric d_2 is CAT(0);

(iv) $\mathcal{K}$ is simply connected and for every vertex $x \in V(\mathcal{K})$, the graph Link(x) is triangle-free.

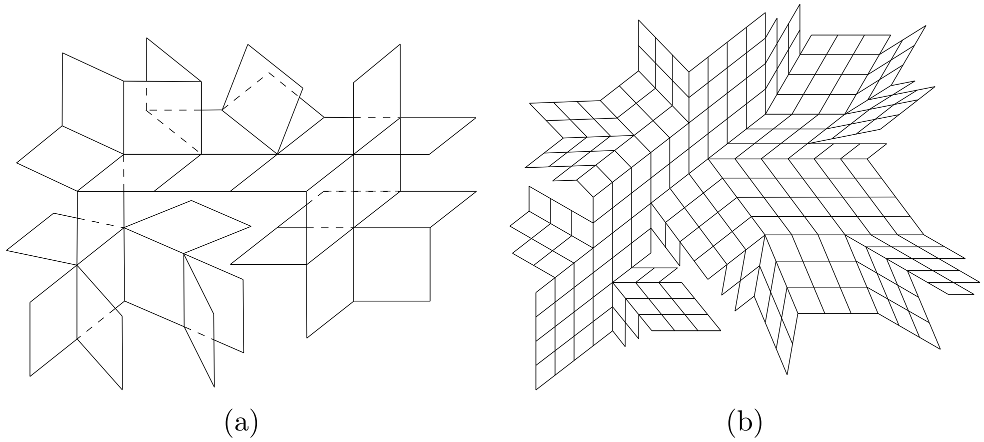


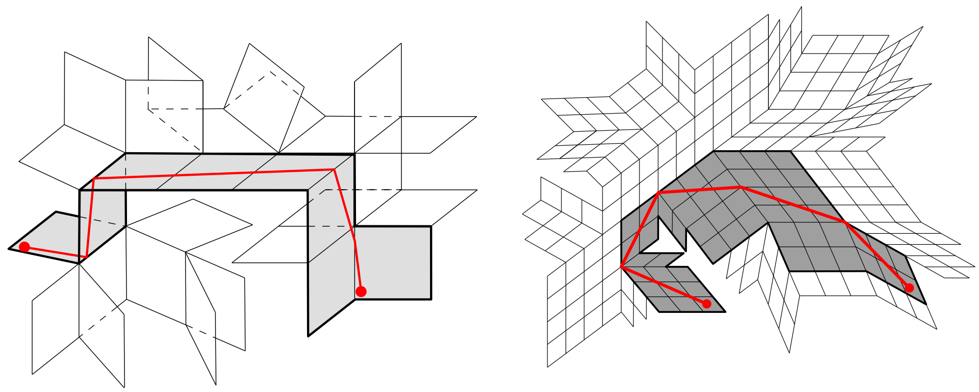
Figure 1: (a) a CAT(0) rectangular complex, (b) a squaregraph.

Typical examples of CAT(0) rectangular complexes (which are illustrated in Fig. 1) are the *squaregraphs* (i.e., the rectangular complexes obtained from the plane graphs in which all inner faces are 4-cycles and all inner vertices have degrees ≥ 4 [4, 16]) and the *ramified rectilinear polygons* (i.e., rectangular complexes endowed with the intrinsic l_1-metric which embeds isometrically into the product of two finite dendrons [5]). As is established in [5], ramified rectilinear polygons are exactly the simply connected rectangular complexes $\mathcal{K}$ in which the graph Link(x) is bipartite for each vertex x of $\mathcal{K}$.

In median graphs, the halfspaces (the convex sets with convex complements) have a special structure and plays an important role. It is well known [7, 37, 38] that median graphs isometrically embed into hypercubes. The isometric embedding of a median graph G into a (smallest) hypercube coincides with the so-called canonical embedding, which is determined by the Djoković-Winkler relation Θ on the edge set of G : two edges uv and wx are Θ-related exactly when $d_G(u,w) + d_G(v,x) \neq d_G(u,x) + d_G(v,w)$; see [24, 32]. For a median graph this relation is transitive and hence an equivalence relation. It is the transitive closure of the "opposite" relation of edges on 4-cycles (i.e., 2-dimensional faces of $\mathcal{K}$): in fact, any two Θ-related edges can be connected by a ladder (viz., the Cartesian product of a path with K_2), and all edges Θ-related to some edge uv constitute a cutset $\Theta(uv)$ of the median graph, which determines one factor of the canonical hypercube [37]. The cutset $\Theta(uv)$ defines two complementary halfspaces (convex sets with convex complements) $W(u,v), W(v,u)$ of G [38, 43], where $W(u,v) = \{x \in X : d(u,x) < d(v,x)\}$ and $W(v,u) = V - W(u,v)$. Conversely, for any pair of complementary halfspaces H_1, H_2 of a median graph G there exists an edge xy such that $W(x,y) = H_1$ and $W(y,x) = H_2$ is the given pair of halfspaces (in fact, all edges belonging to the same equivalence Θ-class as xy define the same pair of complementary halfspaces).

In this paper, we consider the following shortest path problem in CAT(0) rectangular complexes $\mathcal{K}$ endowed with the intrinsic l_2-metric d_2 (we illustrate this formulation in Fig. 2):

Two-point queries: Given two points x, y of $\mathcal{K}$, compute the unique shortest path $\gamma(x,y)$ between x and y in $\mathcal{K}$.



(a) (b)

Figure 2: Two-point shortest path queries.

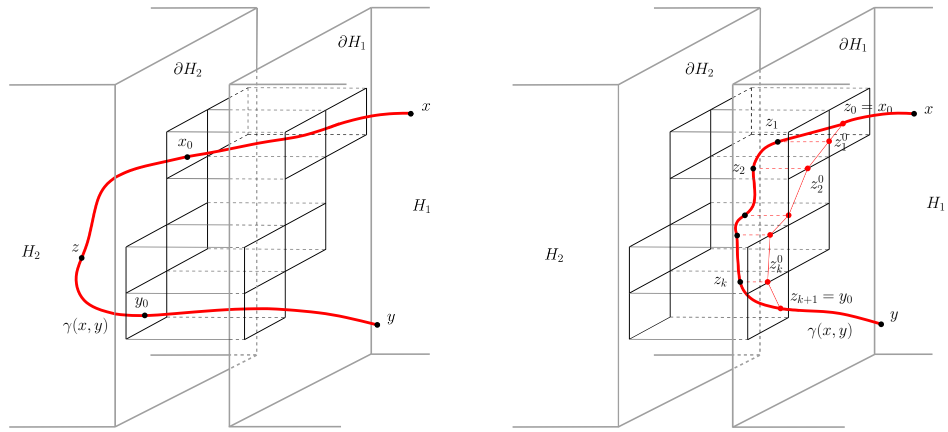
3 Geodesics and graph-intervals

In this section, we show that, given two arbitrary points x, y of a CAT(0) box complex $\mathcal{K}$, the d_2-shortest path $\gamma(x, y)$ is always contained in the subcomplex induced by the graph interval $I(p, q)$ between two vertices p, q belonging to the cells containing x and y, respectively. Moreover, we show that this subcomplex $\mathcal{K}(I(p, q))$ can be unfolded in the k-dimensional space $\mathbb{R}^k$ (where k is the dimension of a largest cell of $\mathcal{K}(I(p, q)))$ in a such a way that the d_2-shortest path between any two points is the same in $\mathcal{K}(I(p, q))$ and in the unfolding of $\mathcal{K}(I(p, q))$.

Proposition 1 *If p and q are two vertices of a CAT(0) box complex $\mathcal{K}$, then $\mathcal{K}(I(p, q))$ is d_2-convex and therefore $\gamma(x, y) \subset \mathcal{K}(I(p, q))$ for any two points $x, y \in \mathcal{K}(I(p, q))$.*

Proof. According to Theorem 2.2, CAT(0) box complexes are exactly the box complexes having median graphs as underlying graphs. Let $G = G(\mathcal{K})$ be the underlying graph of $\mathcal{K}$. Since G is a median graph, the interval $I(p, q)$ is a convex subset (and therefore a gated subset) of G [43]. Additionally, in median graphs each convex set S can be written as an intersection of halfspaces [43] (we will present a simple proof of this fact below). Therefore, it suffices to show that for each halfspace H of the graph G, the subcomplex $\mathcal{K}(H)$ is d_2-convex. Indeed, this will show that $\mathcal{K}(S)$ can be represented as an intersection of d_2-convex sets of $\mathcal{K}$ and therefore that $\mathcal{K}(S)$ itself is d_2-convex.

Now, we show that any convex set S of a median graph G is the intersection of the half-spaces containing S. For this, it suffices to show that for any vertex v of G not belonging to S there exists a pair of complementary halfspaces H_1, H_2 of G such that $v \in H_1$ and $S \subseteq H_2$. Since S is convex and G is median, S is gated [43]. Let u be the gate of v in S (i.e., $u \in I(v, x)$ for each vertex $x \in S$). Let v' be a neighbor of v in the interval $I(v, u)$. Consider the complementary halfspaces $W(v, v')$ and $W(v', v)$ of G defined by the edge vv'. Then obviously $v \in W(v, v')$. On the other hand, since $v' \in I(v, u) \subseteq I(v, x)$ for any vertex $x \in S$, by the definition of $W(v', v)$ we conclude that $x \in W(v', v)$, yielding $S \subseteq W(v', v)$ and concluding the proof.



(a) (b)

Figure 3: To the proof of Proposition 1.

Finally, we will prove now that for any pair of complementary halfspaces H_1, H_2 of G the subcomplexes $\mathcal{K}(H_1)$ and $\mathcal{K}(H_2)$ are d_2-convex. Suppose without loss of generality that H_1 and H_2 are defined by the edges of the equivalence class Θ_i of Θ. The *boundary* ∂H_1 of H_1 consists of all ends of edges of Θ_i belonging to H_1 (the boundary ∂H_2 of H_2 is defined in a similar way). The boundaries ∂H_1 and ∂H_2 induce isomorphic convex, and therefore median, subgraphs of G. Hence $\mathcal{K}(\partial H_1)$ and $\mathcal{K}(\partial H_2)$ are isomorphic CAT(0) subcomplexes of $\mathcal{K}$, which we call *hyperplanes*. Note also that $H_1 \cup \partial H_2$ and $H_2 \cup \partial H_1$ induce convex, and therefore median, subgraphs of G. All edges xy of Θ_i have the same length l_i in $\mathcal{K}$. The CAT(0) subcomplex $\mathcal{K}(\partial H_1 \cup \partial H_2)$ of $\mathcal{K}$ is isomorphic to the CAT(0) box complex $\mathcal{K}_i = \partial H_1 \times [0, l_i]$. We will show that $\mathcal{K}(H_1)$ and $\mathcal{K}(H_2)$ are d_2-convex by induction on the number of vertices of $\mathcal{K}$. Suppose by way of contradiction that $\mathcal{K}(H_1)$ is not d_2-convex. Then there exist two points $x, y \in \mathcal{K}(H_1)$ such that the geodesic $\gamma(x, y)$ has a point which does not belong to $\mathcal{K}(H_1)$. First suppose that $\gamma(x, y)$ contains a point z which belongs to $\mathcal{K}(H_2 \setminus \partial H_2)$; see Fig. 3(a) for an illustration. Then we can find two points $x_0 \in \gamma(x, z) \cap \mathcal{K}(\partial H_2)$ and $y_0 \in \gamma(y, z) \cap \mathcal{K}(\partial H_2)$. Then $z \in \gamma(x_0, y_0)$, showing that $\mathcal{K}(\partial H_2)$ is not d_2-convex in $\mathcal{K}$ and in the CAT(0) subcomplex $\mathcal{K}(H_2)$. Since ∂H_2 is a convex subgraph of G and therefore of the median subgraph $G(H_2)$ of G induced by H_2, we conclude that in the CAT(0) complex $\mathcal{K}(H_2)$ not every convex subgraph induces a d_2-convex subcomplex, contrary to the induction hypothesis.

Therefore, we can assume that the geodesic $\gamma(x, y)$ is entirely contained in the subcomplex of $\mathcal{K}$ induced by $H_1 \cup \partial H_2$; this case is illustrated in Fig. 3(b). Since $\gamma(x, y)$ is a closed set of $\mathcal{K}$, we can find two (necessarily different) points $x_0, y_0 \in \gamma(x, y) \cap \mathcal{K}(\partial H_1)$ such that all points of the geodesic $\gamma(x_0, y_0)$ except x_0 and y_0 all belong to the strip $\mathcal{K}_i$ minus the hyperplane $\mathcal{K}(\partial H_1)$. Let $C_0, C_1, \ldots, C_k$ be the sequence of maximal by inclusion cells of $\mathcal{K}_i$ intersected by $\gamma(x_0, y_0)$ labeled in order in which they intersect $\gamma(x_0, y_0)$ (so that $x_0 \in C_0$ and $y_0 \in C_k$). Let $z_i \in \gamma(x_0, y_0) \cap C_{i-1} \cap C_i$, $i = 1, \ldots, k$, and set $z_0 = x_0, z_{k+1} = y_0$. Then each of the geodesics $\gamma(z_{i-1}, z_i)$ belongs to the cell C_{i-1}, respectively. The intersection C_i^0 of each cell C_i with the hyperplane $\mathcal{K}(\partial H_1)$ is a cell of $\mathcal{K}(\partial H_1)$ and also a facet of C_i. The orthogonal projection π_i of each box C_i $(i = 0, \ldots, k)$ on its facet C_i^0 is a non-expansive map (with respect to the d_2-metric). Notice that $\pi_0(z_0) = z_0 = y_0$ and $\pi_k(z_{k+1}) = z_{k+1} = y_0$. On the other hand, since $z_i \in C_{i-1} \cap C_i$ and the cells C_{i-1} and C_i are axis-parallel boxes, we conclude that the two projections $\pi_{i-1}(z_i)$ and $\pi_i(z_i)$ are one and the same point z_i^0 of the cell $C_{i-1}^0 \cap C_i^0 \subset C_{i-1} \cap C_i$. Consider the path $\gamma^0(x_0, y_0)$ between x_0 and y_0 in the hyperplane $\mathcal{K}(\partial H_1)$ obtained by concatenating the geodesics $\gamma(z_0^0 = x_0, z_1^0), \gamma(z_1^0, z_2^0), \ldots, \gamma(z_k^0, y_0 = z_{k+1}^0)$. Since for each $i = 0, 1, \ldots, k$, the map π_i is non-expansive on C_i, the length of the geodesic $\gamma(z_i^0, z_{i+1}^0)$ is less or equal to the length of the geodesic $\gamma(z_i, z_{i+1})$. Therefore $\gamma^0(x_0, y_0)$ is a geodesic between x_0 and y_0 which is completely contained in the hyperplane $\mathcal{K}(\partial H_1)$. Since $\gamma(x_0, y_0) \cap \mathcal{K}(\partial H_1) = \{x_0, y_0\}$, we conclude that x_0 and y_0 are connected in $\mathcal{K}$ by two different geodesics $\gamma(x_0, y_0)$ and $\gamma^0(x_0, y_0)$, contrary to the assumption that $\mathcal{K}$ is CAT(0). This contradiction establishes that the halfspaces of G induce indeed d_2-convex subcomplexes of $\mathcal{K}$, establishing in particular that $\mathcal{K}(I(p, q))$ is d_2-convex for any two vertices p, q of $\mathcal{K}$. $\square$

Proposition 2 *If x and y are two arbitrary points of a CAT(0) box complex $\mathcal{K}$ and $R(x), R(y)$ are two minimal by inclusion cells of $\mathcal{K}$ containing x and y, respectively, then $\gamma(x, y) \subset \mathcal{K}(I(p, q))$, where p and q are mutually furthest (in the graph $G(\mathcal{K})$) vertices of $R(x)$ and $R(y)$.*

Proof. $R(x)$ and $R(y)$ are the unique cells of least dimension such that x belongs to the relative interior of $R(x)$ and y belongs to the relative interior of $R(y)$. The sets of vertices of $R(x)$ and $R(y)$ are convex, and therefore gated, subsets of G. Let $p \in R(x)$ and $q \in R(y)$ be two mutually furthest vertices of $R(x)$ and $R(y)$, i.e. $d(p, q) = \max\{d(p', q') : p' \in V(R(x)), q' \in V(R(y))\}$, where all distances $d(p', q')$ are computed according to the graph-distance in $G(\mathcal{K})$. Since G is bipartite, the choice of the pair p, q implies that all neighbors of p in the graphic cube $G(R(x))$ must be one step closer to q than p, i.e., all these vertices (we denote this set by A) belong to the interval $I(p, q)$. Analogously, all neighbors of q in $G(R(y))$ (we denote this set by B) also belong to the interval $I(p, q)$. Since $I(p, q)$ is convex and the convex hull of A in G contains the whole graphic cube $G(R(x))$ while the convex hull of B contains $G(R(y))$, we conclude that both $R(x)$ and $R(y)$ belong to the subcomplex $\mathcal{K}(I(p, q))$. Since by Proposition 1 $\mathcal{K}(I(p, q))$ is d_2-convex and $x \in R(x), y \in R(y)$, we conclude that $\gamma(x, y) \subset \mathcal{K}(I(p, q))$. $\square$

The next result shows that the intervals $I(p, q)$ in the CAT(0) box complexes can be unfolded in Euclidean spaces of dimension equal to the topological dimension (i.e., the least dimension of a cell) of $\mathcal{K}(I(p, q))$. Recall that a function $f : X \to X'$ between two metric spaces (X, d) and (X', d') is an *isometric embedding* of X into X' if $d'(f(x), f(y)) = d(x, y)$ for any $x, y \in X$. In this case $Y := f(X)$ is called an *(isometric) subspace* of X'. If a mapping $f : X \mapsto X'$ between two Menger-convex metric spaces (X, d) and (X', d') is such that $f(X)$ is Menger-convex and compact, then $f(X)$ is called an *unfolding* of X in X' if f is an isometric embedding of (X, d) in $(f(X), d^*)$, where d^* is the intrinsic metric on $f(X)$ induced by d'.

Proposition 3 *If p and q are two vertices of a CAT(0) box complex $\mathcal{K}$ and k is the largest dimension of a cell of $\mathcal{K}(I(p,q))$, then there exists an embedding f^* of $\mathcal{K}(I(p,q))$ in the k-dimensional Euclidean space $\mathbb{R}^k$ such that $f^*(\mathcal{K}(I(p,q)))$ is an unfolding of $\mathcal{K}(I(p,q))$.*

Proof. First, we show that the subgraph $G(I(p,q))$ of $G = G(\mathcal{K})$ induced by the interval $I(p,q)$ can be isometrically embedded in the k-dimensional cubical grid $\mathbb{Z}^k = \Pi_{i=1}^k P_i$, where each P_i is the infinite in two directions path. Indeed, intervals $I(p,q)$ of median graphs and median semilattices can be viewed as distributive lattices by setting $x \wedge y = m(p,x,y)$ and $x \vee y = m(q,x,y)$ for any $x,y \in I(p,q)$, where m is the median operator of G [6, 10]. Using the encoding of distributive lattices via closed subsets of a poset due to Birkhoff [9], the famous Dilworth's theorem (the size of a largest antichain of a poset equals to the least size of a decomposition of the poset into chains) [21] implies that any distributive lattice L of breadth k can be embedded as a sublattice of a product of k chains, see [33] or [19] for this interpretation of Dilworth's result (the breadth of a distributive lattice L is equal to the largest out- or in-degree of a vertex in the covering graph of L). Larson [33] showed that the resulting embedding can be chosen to preserve the covering relation, i.e. to be a graph embedding. Recently, using the same tools, Cheng and Suzuki [19] noticed that the embedding can be selected to be an isometric embedding of the covering graph of a distributive lattice of breadth k in the product of k chains (note that Eppstein [23] showed how to decide in polynomial time if a graph G isometrically embeds into the product of k chains).

Therefore, it remains to show that the largest out-degree or in-degree of $G(I(p,q))$ equals to the dimension of a largest cube of $G(I(p,q))$. For this, it suffices to show that if $v \in I(p,q)$, then any m neighbors $y_1, y_2, \ldots, y_m \in I(v,q) \subseteq I(p,q)$ of v define an m-cube $C_m \subseteq I(v,q)$. We proceed by induction on m. Denote by C' the $(m-1)$-cube induced by the vertices $y_1, \ldots, y_{m-1}$ (which exists because of the induction assumption). Let z_i be the median of the triplet y_i, y_m, q. Then $z_1, \ldots, z_{m-1}$ are pairwise different (because median graphs are $K_{2,3}$-free), are all adjacent to y_m and all belong to the interval $I(y_m, q)$. Therefore, by induction hypothesis, $z_1, \ldots, z_{m-1}$ induce an $(m-1)$-cube $C'' \subseteq I(y_m, q)$. Then it can be easily shown that each vertex of C' is adjacent to a unique vertex of C'' and that this adjacency relation induces an isomorphism between the cubes C' and C''. Hence $C' \cup C''$ is an m-dimensional cube. Thus indeed $G(I(p,q))$ can be isometrically embedded in the k-dimensional grid $\mathbb{Z}^k$. Denote by f such an embedding. To transform f into an unfolding of $\mathcal{K}(I(p,q))$ in $\mathbb{R}^k$, we simply transform the uniform cubical grid $\mathbb{Z}^k$ into a non-uniform one: notice that all edges of the same equivalence class Θ_i of $G(I(p,q))$ are mapped by f to one and the same edge e of some path P_j. If the edges of Θ_i all have length l_i, then we simply assign length l_i to the edge e of P_j. Denote the resulting paths by $P_1^*, \ldots, P_k^*$ and notice that after this scaling the previous embedding induce an embedding f^* of the graph $G(I(p,q))$ weighted by the length of edges in $\mathcal{K}$ into the grid $\Pi_{i=1}^k P_i^*$. We can extend in a natural way f^* to an embedding of $\mathcal{K}(I(u,v))$ into $\mathbb{R}^k = ||\Pi_{i=1}^k P_i^*||$: for a cell R of $\mathcal{K}(I(u,v))$, $f^*(R)$ is the cell induced by the images under f^* of the vertices of R. Let $f^*(\mathcal{K}(I(p,q)))$ denote the box complex consisting of the images of all cells of $\mathcal{K}(I(p,q))$. Since each path between two points x,y of $\mathcal{K}(I(p,q))$ is mapped to a path of the same length of $f^*(\mathcal{K}(I(p,q)))$ between $f^*(x)$ and $f^*(y)$, we obtain the desired unfolding of $\mathcal{K}(I(p,q))$ in the k-dimensional Euclidean space. □

For efficient (but nonlinear) algorithms for isometric embeddings of median graphs into cubical grids of least dimension, see the recent paper by Cheng [18].

4 Two-point shortest path queries

In this section, we present the detailed description of the algorithm for answering two-point shortest path queries in CAT(0) rectangular complexes and of the data structure $\mathcal{D}$ used in this algorithm. First we show that $\mathcal{K}(I(p,q))$ always can be unfolded in the plane as a chain of monotone polygons, which we will denote by $P(I(p,q))$, and we show how to compute this unfolding efficiently. Therefore, to compute $\gamma(x,y)$, we triangulate each monotone polygon of $P(I(p,q))$ and compute in linear time the shortest path $\gamma^*(x,y)$ in $P(I(p,q))$ between the images of x and y (we denote them also by x,y) using the algorithm of Lee and Preparata [34] and return as $\gamma(x,y)$ the preimage of $\gamma^*(x,y)$. As a preprocessing step, we design a data structure $\mathcal{D}$ allowing for each query x,y to efficiently retrieve the boundary of an interval $I(p,q)$ such that $x,y \in \mathcal{K}(I(p,q))$ and x,p and y,q belongs to common rectangular cells, respectively (in time proportional to the distance $d(p,q)$ between p and q in $G(\mathcal{K})$).

4.1 The unfolding of $\mathcal{K}(I(p,q))$ in $\mathbb{R}^2$

From Proposition 3, we know that for any two vertices p and q of a CAT(0) rectangular complex $\mathcal{K}$, the graph $G(I(p,q))$ is isometrically embeddable in $\mathbb{Z}^2$ and consequently the subcomplex $\mathcal{K}(I(p,q))$ can be unfolded in $\mathbb{R}^2$. We will show how to compute such an unfolding efficiently. Denote by $P(I(p,q))$ the image of $\mathcal{K}(I(p,q))$ under an unfolding f (which we know to exist). Let $B_1,\ldots,B_m$ be the 2-connected components (alias blocks) of the graph $G(I(p,q))$. From the definition of $I(p,q)$, it follows that $B_1,\ldots,B_m$ define a chain of blocks, i.e., if $s_1 \in B_1 \cap B_2,\ldots,s_{m-1} \in B_{m-1} \cap B_m$ are the articulation vertices of $G(I(p,q))$ and $s_0 = p, s_m = q$, then $B_i = I(s_{i-1},s_i)$, $i = 1,\ldots,m$, and all vertices $s_1,\ldots,s_{m-1}$ belong to all shortest paths between p and q. In the squaregraph $G(I(p,q))$ each hyperplane is a convex path and in any isometric embedding of $G(I(p,q))$ in $\mathbb{Z}^2$ the image of this path is a horizontal or a vertical path. Therefore $P(I(p,q)) = \cup_{i=1}^{m} P_i$ is horizontally and vertically convex and consists of a chain of monotone polygons $P_1 = f(\mathcal{K}(B_1)),\ldots,P_m = f(\mathcal{K}(B_m))$. Up to translations, rotations, and symmetries, each block B_i has a unique isometric embedding in $\mathbb{Z}^2$, whence each $\mathcal{K}(B_i)$ has a unique unfolding in the plane. We will define the *boundary* $\partial G(I(p,q))$ of $G(I(p,q))$ (or of $\mathcal{K}(I(p,q)))$ as the subgraph induced by all edges of $G(I(p,q))$ which are mapped to the boundary $\partial P(I(p,q))$ of $P(I(p,q))$ (since $G(I(p,q))$ is a squaregraph, this definition is equivalent to the definition of a boundary of a squaregraph given in [4]). Given two arbitrary points $x,y \in \mathcal{K}(I(p,q))$, it is a well-known property of simple polygons that all vertices of the (Euclidean) shortest path in $P(I(p,q))$ between $f(x)$ and $f(y)$ are vertices of the boundary of $P(I(p,q))$(except $f(x)$ and $f(y)$). Therefore, instead of defining the image of the whole subcomplex $\mathcal{K}(I(p,q))$ under an unfolding map f, it suffices to define only the

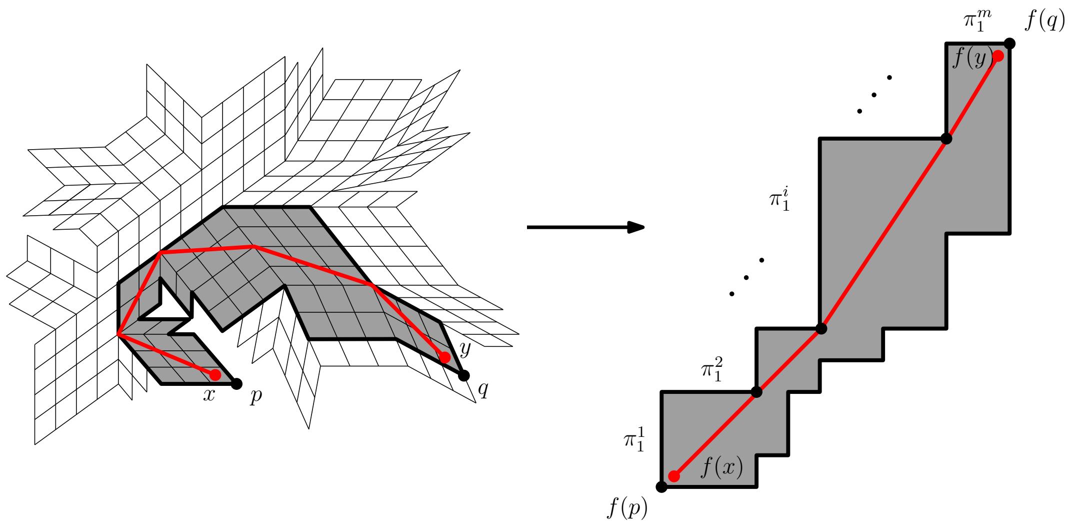


Figure 4: An unfolding of $\mathcal{K}(I(p,q))$.

image under f of its boundary $\partial G(I(p,q))$ (in this case, we will speak about the unfolding of $\partial G(I(p,q))$).

Proposition 4 *If p and q are two vertices of a CAT(0) rectangular complex $\mathcal{K}$, then $\mathcal{K}(I(p,q))$ can be unfolded in the Euclidean plane as a chain of monotone polygons and this embedding can be constructed in $O(|I(p,q)|)$ time if $I(p,q)$ is given. Moreover, if the boundary $\partial G(I(p,q))$ is given together with the degrees $deg_0(z)$ in $G(I(p,q))$ of all of its vertices z, then an unfolding of $\partial G(I(p,q))$ can be constructed in $O(d(p,q))$ time, where $d(p,q)$ is the distance between the vertices p and q in the graph $G(\mathcal{K})$.*

Proof. Suppose that either $I(p,q)$ or its boundary $\partial G(I(p,q))$ is given and we will show how to isometrically embed $G(I(p,q))$ in $\mathbb{Z}^2$ (the passage from the uniform to the non-uniform grid is the same as in the general case described in the proof of Proposition 3). To embed $G(I(p,q))$, it suffices to detect in linear time the 2-connected components of $G(I(p,q))$ or of its boundary $\partial G(I(p,q))$, to embed each block B_i or its boundary ∂B_i, and to compose these embeddings into a single one by identifying the images of common articulation vertices of blocks (see the discussion preceding this proposition). Therefore we can suppose without loss of generality that $G(I(p,q))$ is 2-connected. Then, up to translations, symmetries, and rotations by 90°, $G(I(p,q))$ has a unique isometric embedding f into the square grid and $\mathcal{K}(I(p,q))$ is embedded in $\mathbb{R}^2$ as a rectilinear polygon $P = P(I(p,q))$ which is monotone with respect to the coordinate axes. Each vertex of P (including $f(p)$ and $f(q)$) is either a *convex vertex* (i.e., the interior angle of P between its two incident edges is equal to 90°) or a *reflex vertex* (i.e., the interior angle between its two incident edges is equal to 270°). Convex vertices of P are exactly the images of vertices of degree 2 of $G(I(p,q))$ (in particular, $f(p)$ and $f(q)$ are convex vertices of P), while reflex vertices are images of vertices of degree 4 lying on the boundary of $G(I(p,q))$. Note also that the vertices of degree 3 of $\partial G(I(p,q))$ are mapped to points lying on sides of P.

Let π_1 and π_2 denote the two disjoint shortest (p,q)-paths constituting the boundary of $G(I(p,q))$. To find the polygon $P(I(p,q))$ which is the image of $\mathcal{K}(I(p,q))$ under an isometric embedding of $G(I(p,q))$ in $\mathbb{Z}^2$, it suffices to find the images of π_1 and π_2 under such an embedding. For this, first we scan π_1 and π_2 in order to detect the convex and the reflex vertices of each of them (note that the convex and the reflex vertices on each of these two paths alternate). Suppose that convex and reflex vertices subdivide π_1 into the subpaths $\pi_1^1,\ldots,\pi_1^m$. Then we define the image of p to be the point $(0,0)$, draw the image of π_1^1 as a vertical path with one end at $(0,0)$ and having length equal to the length of π_1^1, then draw the image of the path π_1^2 as a horizontal path with the beginning at the point where the former path ended and of length equal to the length of π_1^2, and so on, on step i we draw the image of the current path π_1^i to be orthogonal to the image of the previous path π_1^{i-1} (the direction of π_1^i depends of whether π_1^{i-1} and π_1^i share a convex or a reflex vertex). To draw the images of the subpaths $\pi_2^1,\ldots,\pi_2^{m'}$ of π_2, we proceed in the same way but we start by drawing the image of π_2^1 as a horizontal path. This embedding of $\partial G(I(p,q))$ extends in a natural way to an embedding of $G(I(p,q))$: the image of each vertex v lying on the convex path (hyperplane) with ends $u'\in\pi_1$ and $u''\in\pi_2$ is a vertex $f(u)$ of the horizontal or vertical path between $f(u')$ and $f(u'')$ and lying on distance $d(v,u')$ from $f(u')$ and $d(v,u'')$ from $f(u'')$. Notice that f is an isometric embedding of $G(I(p,q))$ into the grid $\mathbb{Z}^2$ because, up to 90°'s rotations, there exists a unique isometric embedding in which p is mapped to $(0,0)$ and this embedding necessarily satisfies the properties of f (f itself is defined in the canonical way). □

4.2 The data structure $\mathcal{D}$ and the computation of $\partial G(I(p,q))$

In this subsection, we design the data structure $\mathcal{D}$ for general CAT(0) rectangular complexes, for ramified rectilinear polygons, and for squaregraphs, allowing quickly to compute for two arbitrary vertices p,q of $\mathcal{K}$ the boundary paths π_1 and π_2 of $G(I(p,q))$ and the degrees $deg_0(z)$ of all vertices $z\in\pi_1\cup\pi_2=\partial(G(I(p,q)))$. The main requirement to $\mathcal{D}$ is the trade-off between the space occupied by $\mathcal{D}$ and the time for computing π_1 and π_2. Further, we will assume that n denotes the number of vertices of $\mathcal{K}$. Notice that any CAT(0) rectangular complex contains $O(n)$ edges and faces. Indeed, $|E(\mathcal{K})|\le 2n$ and $|F(\mathcal{K})|\le n$, because any cube-free median graph G contains a vertex w of degree at most 2 (if the degree of w is 2, then it belongs to a unique rectangular cell of $\mathcal{K}$, otherwise if the degree of w is 1, then w is incident to a unique edge) and, removing w, the resulting graph G' is also cube-free and median. As is shown in [5], as w one can pick any furthest vertex from a given arbitrary vertex. Since G' contains $n-1$ vertices, $|E(\mathcal{K})|-1$ or $|E(\mathcal{K})|-2$ edges, and $|F(\mathcal{K})|$ or $|F(\mathcal{K})|-1$ faces, the required inequalities for G follows by applying induction assumption to G'.

CAT(0) rectangular complexes. In case of general CAT(0) rectangular complexes, using Breadth-First-Search (BFS), first we compute the distance matrix D of $G(\mathcal{K})$. Additionally, running BFS starting from any vertex u of $V(\mathcal{K})$, for each vertex v we compute the list of neighbors $L_u(v)$ of v in the interval $I(u,v)$ (these are exactly the neighbors of v which has been labeled by BFS before v). Notice that each list $L_u(v)$ contains one or two vertices. Now, $\mathcal{D}$

includes the distance matrix D of the underlying graph $G(\mathcal{K})$ and the lists $L_u(v), u, v \in V(\mathcal{K})$. $\mathcal{D}$ requires $O(n^2)$ space and can be constructed in $O(|V(\mathcal{K})||E(\mathcal{K})|) = O(n^2)$ time.

Now, we will show how to use $\mathcal{D}$ to construct the boundary paths π_1 and π_2 of $G(I(p,q))$. Before describing the algorithm, first notice that the vertices $z \in \partial G(I(p,q))$ with $deg_0(z) = 4$ are exactly the reflex vertices of $\partial G(I(p,q))$. If say z is a reflex vertex of π_1, then one can easily see that if x is the neighbor of z in π_1 located one step closer to p, then necessarily $L_q(x) = \{z\}$. To build the paths π_1 and π_2, we initialize $\pi_1 := \{p\} =: \pi_2$. If $L_q(p)$ consists of a single vertex p', then we set $\pi_1 := \pi_1 \cup \{p'\}$ and $\pi_2 := \pi_2 \cup \{p'\}$. If $L_q(p)$ consists of two different vertices p' and p'', then set $\pi_1 := \pi_1 \cup \{p'\}$ and $\pi_2 := \pi_2 \cup \{p''\}$. Now suppose that after k steps, x is the last vertex inserted in π_1 and y is the last vertex inserted in π_2, and we have to compute the neighbor x' of x in π_1 and the neighbor y' of y in π_2. Notice that x and y have the same distance k to p. If $x = y$ (i.e, this is an articulation vertex of $G(I(p,q))$ and $\partial G(I(p,q))$), then we proceed the vertex $x = y$ in the same way as the starting vertex p. Now, suppose that $x \neq y$. If $L_q(x)$ consists of a single vertex a, then set $\pi_1 := \pi_1 \cup \{a\}$ and $x' := a$. Additionally, if $L_q(y) = \{a\}$, then set $\pi_2 := \pi_2 \cup \{a\}$ and $y' = a$. On the other hand, if $L_q(x) = \{a\}$ and $L_q(y) = \{a, b\}$, then set $\pi_2 := \pi_2 \cup \{b\}$ and $y' = b$. Notice that in this case, x and b cannot be adjacent, otherwise the vertices a, b, x, y, and the median of the triplet x, y, p induce a forbidden $K_{2,3}$. The case when $L_q(x) = \{a\}$ and $L_q(y) = \{b, c\}$ can be treated as the remaining case below. Now suppose that $L_q(x)$ consists of two vertices a and b (the case when $|L_q(y)| = 2$ is analogous). Since one of these vertices belongs to the boundary of $G(I(p,q))$ and $|L_q(x)| = 2$, from what has been noticed above, the case $deg_0(a) = deg_0(b) = 4$ is impossible. If $deg_0(a) \leq 3$ and $deg_0(b) = 4$, then set $\pi_1 := \pi_1 \cup \{a\}$ and $x' = a$. Finally suppose that $deg_0(a) \leq 3$ and $deg_0(b) \leq 3$. Then, if $y \notin L_p(a)$, set $\pi_1 := \pi_1 \cup \{a\}$, $x' = a$ and $\pi_2 := \pi_1 \cup \{b\}$, $x' = b$. Since $\partial G(I(p,q))$ has one or two vertices located at distance $k+1$ from p, in the previous case necessarily y must be adjacent to b, showing why we can add the vertex b to π_2. On the other hand, y cannot be adjacent to both a and b, otherwise the vertices a, b, x, y, and the median of x, y, p induce a forbidden $K_{2,3}$. As a result, with a data structure $\mathcal{D}$ of size $O(n^2)$, for each pair of vertices p, q, we can construct the boundary of $G(I(p,q))$ in $O(d(p,q))$ time. The degrees $deg_0(z)$ of vertices $z \in \pi_1 \cup \pi_2$ in the graph $G(I(p,q))$ can be easily computed in total $O(d(p,q))$ time by setting $deg_0(z) := |L_p(z)| + |L_q(z)| \leq 4$.

Ramified rectilinear polygons. In case of ramified rectilinear polygons $\mathcal{K}$, the data structure $\mathcal{D}$ mainly consists of an isometric embedding of $G(\mathcal{K})$ into the Cartesian product of two trees T_1, T_2 and a data structure for nearest common ancestor queries in trees build on each of the tree-factors T_1 and T_2. As was shown in [5], the trees T_1 and T_2 and the isometric embedding of $G := G(\mathcal{K})$ in $T_1 \times T_2$ can be obtained in the following way. First we construct the equivalence classes $\Theta_1, \ldots, \Theta_m$ of the Djoković-Winkler relation Θ on the edge set of G. As we noticed already, Θ is the transitive closure of the "opposite" relation of edges of rectangular cells of $\mathcal{K}$, and therefore the equivalence classes of Θ can be easily constructed in total $O(|F(\mathcal{K})| + |E(\mathcal{K})| + |V(\mathcal{K})|) = O(n)$ time. Simultaneously with Θ, we can construct the *incompatibility graph* $\mathrm{Inc}(G)$ of G: the equivalence classes $\Theta_1, \ldots, \Theta_m$ are the vertices of $\mathrm{Inc}(G)$ and two equivalence classes Θ_i and Θ_j define an edge of $\mathrm{Inc}(G)$ if and only if there exists a rectangular face of $\mathcal{K}$ in which two opposite edges belong to Θ_i and two other opposite edges belong to Θ_j (the definition of $\mathrm{Inc}(G)$ is different but equivalent to that given in [5]). Clearly, $\mathrm{Inc}(G)$ can be constructed in linear time: it suffices to consider each of the rectangular faces of $\mathcal{K}$ and to define an edge between the two equivalence classes sharing edges wit this face. The graph $\mathrm{Inc}(G)$ is bipartite [5], moreover a coloring of $\mathrm{Inc}(G)$ in two colors define the two tree-factors T_1 and T_2. The trees T_i ($i = 1, 2$) are obtained from G by collapsing all edges colored with a color different from i. Equivalently, to construct T_i, we remove the edges (but keep their ends) colored with color i and compute the connected components of the resulting graph G_i. Each connected component C of G_i defines a vertex of T_i and two connected components C', C'' of G_i define an edge $C'C''$ of T_i if there exists an edge of G (necessarily colored i) with one end in C' and another end in C''. Notice that each edge of T_1 or T_2 is labeled by a different equivalence class of Θ. Now, if u is an arbitrary vertex of G and $C_1(u)$ and $C_2(u)$ are the connected components of G_1 and G_2, respectively, containing the vertex u, then the couple $f(u) = (C_1(u), C_2(u))$ are the coordinates of u in the isometric embedding f of G in $T_1 \times T_2$. The trees T_1, T_2 and the coordinates of the vertices of G in $T_1 \times T_2$ can be defined in total $O(n)$ time. We preprocess the trees T_1 and T_2 in linear time to answer in constant time lowest common ancestor queries [29]. Additionally, for each vertex v we define the sorted list $Q(v)$ of the equivalence classes of Θ to which belong the edges incident to v. These lists $Q(v), v \in V(\mathcal{K})$, occupy total linear space. These are the three constituents of the data structure $\mathcal{D}$.

Now, we will show how to use $\mathcal{D}$ to construct the boundary paths π_1 and π_2 of $G(I(p,q))$ and the degrees $deg_0(z)$ in $G(I(p,q))$ of their vertices z. First, given the coordinates $f(p) = (C_1(p), C_2(p))$ and $f(q) = (C_1(q), C_2(q))$ of p and q in $T_1 \times T_2$, using two lowest common ancestor queries, one for $C_1(p)$ and $C_1(q)$ in T_1 and the second one for $C_2(p)$ and $C_2(q)$ in T_2, we can compute the path P_1 connecting $C_1(p)$ and $C_1(q)$ in T_1 and the path P_2 connecting $C_2(p)$ and $C_2(q)$ in T_2 in time proportional to the number of edges on these paths. Then $G(I(p,q))$ can be isometrically embedded in the Cartesian product $P_1 \times P_2$. We will call P_1 and P_2 vertical and horizontal paths, respectively. We also suppose that $f(p)$ and $f(q)$ are respectively the lowest leftmost and the upper rightmost corners of $P_1 \times P_2$. We start by setting $\pi_1 := \{p\} =: \pi_2$ and we will construct π_1 and π_2 in such a way that π_1 is the upper path and π_2 is the lower path of the embedding of $G(I(p,q))$ in $P_1 \times P_2$. Let x be the last vertex of π_1 and we will show how to define the next vertex x' of π_1. Let $f(x) = (C_1(x), C_2(x))$. Suppose that the next edge of P_1 incident to $C_1(x)$ is labeled by Θ_i and the next edge of P_2 incident to $C_2(x)$ is labeled by Θ_j. Using binary search on the sorted list $Q(x)$ we can decide in $O(\log(deg(x)))$ time if x has an incident edge belonging to the equivalence class Θ_i and/or an incident edge belonging to Θ_j. If x has an incident edge e from Θ_i, then we set x' to be the end-vertex of e which is different from x (notice that $C_2(x') = C_2(x)$). In this case, the path π_1 goes vertically. Otherwise, if no edge of Θ_i is incident to x, then necessarily there is an edge e' of Θ_j incident to x and, in this case, we set x' to be the end-vertex of e' which is different from x. Then $C_1(x') = C_1(x)$ and the path π_1 goes horizontally. Analogously, if y is the last vertex of π_2, in order to define the next vertex y' of π_2, we consider the labels Θ_i and Θ_j of the next edges of the paths P_1 and P_2, respectively, and using binary search on

$Q(y)$ we decide if y has an incident edge belonging to Θ_j and/or an incident edge belonging to Θ_i. If y has an incident edge e from Θ_j, then we set y' to be the other end-vertex of e and in this case the path π_2 goes horizontally. Otherwise, if no edge of Θ_j is incident to y, then there is an edge e' of Θ_i incident to y and in this case we set y' to be the end-vertex of e' different from x and the path P_2 goes vertically. Notice also that in a similar way we can compute in $O(\log(deg(z)))$ time the degree $deg_0(z)$ in $G(I(p,q))$ of each vertex $z \in \pi_1 \cup \pi_2$. As a consequence, the paths π_1, π_2 and the degrees $deg_0(z), z \in \pi_1 \cup \pi_2$ can be computed in total $O(d(p,q)\log\Delta)$ time, where Δ is the maximum degree of a vertex of $G(\mathcal{K})$.

Squaregraphs. Finally, in the case of squaregraphs, as a data structure $\mathcal{D}$ we will take an encoding of plane graphs defining polygonal complexes of nonpositive curvature presented in [17]. Let $G = G(\mathcal{K})$. The data structure from [17] uses vertex labels of size $O(\log^2 n)$ bits (for each vertex of G it uses a label consisting of $O(\log n)$ integers of length at most $\log n$) and allows for each pair u, v of vertices of G to compute in constant time the distance $d(u,v)$ between u and v in G and a neighbor u' of u lying on a shortest path between u and v. Additionally, we suppose that the plane graph G is represented as a planar subdivision in the form of a doubly-connected edge list [20]. Each edge of G belongs to one or two rectangular faces and, using this representation, the vertices and the edges belonging to these faces can be listed in constant time.

Now, given two vertices p and q, in order to construct the boundary paths π_1 and π_2 of $G(I(p,q)$ we proceed in the following way. The algorithm of [17] returns in $O(1)$ time a neighbor p' of p in the interval $I(p,q)$. Necessarily, p' is a vertex of $\partial G(I(p,q))$. Without loss of generality, we assign p' to the path π_1. To find the neighbor p'' of p in π_2 we consider the edges incident to p in one or two faces containing the edge pp'. For each of the endvertices of these edges which are different from p we compute in constant time its distance to q. If one of these vertices is closer to q than p, then we denote it by p'' and insert in π_2. Otherwise, we set $p'' := p'$. Now suppose that x is the last vertex of π_1 and that x_0 is the vertex preceding x in π_1. Suppose also that $deg_0(x_0)$ and the neighbors of x_0 in $G(I(p,q))$ has been already computed. We will show now how to compute in constant time $deg_0(x)$, the neighbors of x in $G(I(p,q))$, and the next neighbor x' of x in π_1. For this, we consider the faces of $\mathcal{K}$ incident to the edge x_0x and performing in constant time distance queries from [17], we compute which vertex x_1 adjacent to x from these faces belongs to $I(p,q)$ (since $I(p,q)$ is convex and the edge x_0x belongs to the boundary of $G(I(p,q))$, only one such vertex can belong to $I(p,q)$). Then we consider the face of $\mathcal{K}$ incident to xx_1 and not containing x_0 and test if the second neighbor of x in this face belongs to $I(p,q)$. If not, then we set $deg_0(x) := 2$ and $x' := x_1$. Otherwise, if this vertex belongs to $I(p,q)$, then denote it by x_2 and test if the neighbor of x in the second face incident to the edge xx_2 belongs to $I(p,q)$. Notice that after a constant number of such tests, we will find a neighbor x_i of x such that the neighbor x_{i+1} of x in the face of $\mathcal{K}$ containing xx_i and not containing x_{i-1} does not belong to $I(p,q)$ or does not exist (because this face does not exist). In this case, we return $x_0, x_1, \ldots, x_i$ as the neighbors of x in $G(I(p,q))$ and set $deg_0(x) := i+1$ and $x' := x_i$. Thus, using the data structure $\mathcal{D}$ of size $O(n \log n)$ we can construct the boundary path π_1 and compute the degrees in $G(I(p,q))$

of its vertices in total $O(d(p,q))$ time. The path π_2 can be constructed in a similar way and within the same time bounds.

4.3 The algorithm

Summarizing the results of the previous subsections, we are ready to present the main steps of the algorithm for answering shortest path queries in CAT(0) rectangular complexes, ramified rectilinear polygons, and squaregraphs.

Algorithm TWO-POINT SHORTEST PATH QUERIES

Input: A CAT(0) rectangular complex $\mathcal{K}$, a data structure $\mathcal{D}$, and two points $x, y \in \mathcal{K}$

Output: The shortest path $\gamma(x,y)$ between x and y in $\mathcal{K}$

1. Given the rectangular faces containing the points x and y, compute the vertices p, q of $\mathcal{K}$ such that $x, y \in \mathcal{K}(I(p,q))$.
2. Using the data structure $\mathcal{D}$, compute the boundary $\partial G(I(p,q))$ and the degrees $deg_0(z)$ in $G(I(p,q))$ of the vertices z of $\partial G(I(p,q))$.
3. Using the algorithm described in the proof of Proposition 4, compute an unfolding f of $\partial G(I(p,q))$. Let $P(I(p,q))$ denote the chain of monotone polygons bounded by $f(\partial G(I(p,q)))$.
4. Locate $f(x)$ and $f(y)$ in $P(I(p,q))$.
5. Using the algorithm for triangulating monotone polygons (see, for example, [20]), triangulate each monotone polygon constituting a block of $P(I(p,q))$.
6. In the triangulated polygon $P(I(p,q))$ run the algorithm of Lee and Preparata [34] and return the shortest path $\gamma^*(f(x), f(y)) = (f(x), z_1, \ldots, z_m, f(y))$ between $f(x)$ and $f(y)$ in $P(I(p,q))$, where $z_1, \ldots, z_m$ are all vertices of $P(I(p,q))$.
7. Return $(x, f^{-1}(z_1), \ldots, f^{-1}(z_m), y)$ as the shortest path $\gamma(x,y)$ between the points x and y.

It remains to precise how to implement the steps 1 and 2 of the algorithm. For step 1, given two points x and y of $\mathcal{K}$, we are also given two rectangular faces $R(x)$ and $R(y)$ containing x and y. Then using the distance matrix D for CAT(0) rectangular complexes, the coordinates of the embedding in the case of ramified rectilinear polygons, and the distance queries from [17] for squaregraphs, we can compute two furthest vertices p and q, where p is a vertex of $R(x)$ and q is a vertex of $R(y)$. This takes constant time because we take the maximum of a list of 16 distances between the vertices of $R(x)$ and $R(y)$. In case of ramified rectilinear polygons, to compute the distance $d(u,v)$ in constant time, it suffices in each tree T_i $(i = 1,2)$ to keep in $\mathcal{D}$ the distance $d_{T_i}(C, R_i)$ in T_i from each vertex C to the root R_i of T_i. Now, if $f(u) = (C_1(u), C_2(u))$ and $f(v) = (C_1(v), C_2(v))$, then we compute the lowest common ancestor C_1 of $C_1(u)$ and $C_1(v)$ in the tree T_1, the lowest common ancestor C_2 of $C_2(u)$ and $C_2(v)$ in T_2, and return as $d(u,v)$ the value $(d_{T_1}(C_1(u), R_1) + d_{T_1}(C_1(v), R_1) - 2d_{T_1}(C_1, R_1)) + (d_{T_2}(C_2(u), R_2) + d_{T_2}(C_2(v), R_2) - 2d_{T_2}(C_2, R_2))$. Notice also that the step 4 of the algorithm also requires constant time: having the coordinates of x in $R(x)$ and of y in $R(y)$, since $R(x)$ is the unique rectangular face incident to p in $\mathcal{K}(I(p,q))$ and $R(y)$ is the unique face incident to q, we can easily locate the images of $R(x)$ and $R(y)$ in the polygon $P(I(p,q))$. Summarizing, here is the main result of this paper:

Theorem 4.1 *Given a CAT(0) rectangular complex $\mathcal{K}$ with n vertices, one can construct a data structure $\mathcal{D}$ of size $O(n^2)$ so that, given any two points $x, y \in \mathcal{K}$, we can compute the shortest l_2-path $\gamma(x, y)$ between x and y in $O(d(p, q))$ time, where p and q are vertices of two faces of $\mathcal{K}$ containing the points x and y, respectively, such that $\gamma(x, y) \subset \mathcal{K}(I(p, q))$ and $d(p, q)$ is the distance between p and q in the graph $G(\mathcal{K})$. If $\mathcal{K}$ is a ramified rectilinear polygon, then one can construct a data structure $\mathcal{D}$ of optimal size $O(n)$ and answer two-point shortest path queries in $O(d(p, q) \log \Delta)$ time, where Δ is the maximal degree of a vertex of $G(\mathcal{K})$. Finally, if $\mathcal{K}$ is a squaregraph, then one can construct a data structure $\mathcal{D}$ of size $O(n \log n)$ and answer two-point shortest path queries in $O(d(p, q))$ time.*

Open questions: (1) We do not know how to design a subquadratic data structure $\mathcal{D}$ allowing to perform two-point shortest path queries in CAT(0) rectangular complexes in $O(d(p, q))$ time or how to use the encoding provided by the isometric embedding of ramified rectilinear polygons into products of two trees to remove the logarithmic factor in the query time.

(2) It will be interesting to generalize our algorithmic results (using Propositions 1-3) to all CAT(0) box complexes, in particular to 3-dimensional CAT(0) box complexes.

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