

Pareto envelopes in \mathbb{R}^3 under l_1 and l_∞ distance functions

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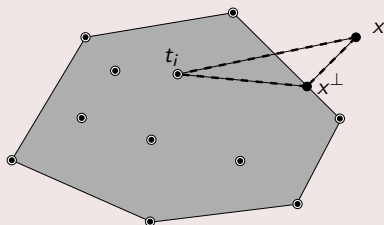


Kuhn's characterization of convex hulls

Let $T = \{t_1, \dots, t_n\}$ be a set of points in \mathbb{R}^m .

Theorem (H. Kuhn, 1967)

$$\text{conv}(T) = \{x \in \mathbb{R}^m : x \text{ is not dominated}\}.$$



$$d(x^\perp, t_j) < d(x, t_j) \text{ for all } x \notin \text{conv}(T) \text{ and all } i = 1, \dots, n$$

Pareto optimality

Pareto optimality : Given a vector function $\mathbf{f} = (f_1, \dots, f_n)$ defined on a set X , a point $y \in X$ is **dominated** by a point $x \in X$ if $f_i(x) \leq f_i(y)$ for all $i \in \{1, \dots, n\}$ and $f_j(x) < f_j(y)$ for some index j . The non-dominated points of X are called the **Pareto optima** of \mathbf{f} .

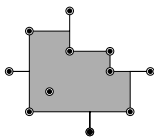
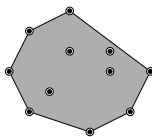
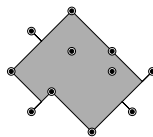
Pareto envelope : Given a metric d on $\mathbb{R}^m (= X)$ and a set of n terminals $T = \{t_1, \dots, t_n\}$, the set of Pareto optima with respect to the vector objective function $\mathbf{d}(x) = (d(t_1, x), \dots, d(t_n, x))$ is called the **Pareto envelope** of T and is denoted by $\mathcal{P}_d(T)$. The set $\mathcal{P}_d^0(T)$ of all points $p \in \mathcal{P}_d(T)$ such that $\mathbf{d}(p) \neq \mathbf{d}(q)$ for any other point q is called the **strict Pareto envelope**.

Our goal : Characterize and efficiently construct the Pareto envelopes $\mathcal{P}_{d_1}(T)$, $\mathcal{P}_{d_\infty}(T)$ and their strict counterparts $\mathcal{P}_{d_1}^0(T)$, $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 endowed with ℓ_1 and ℓ_∞ distances.

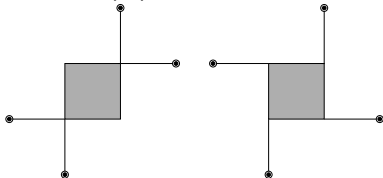


Pareto envelopes in the plane

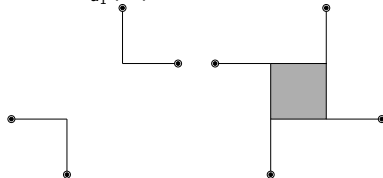
For the same set of terminals T

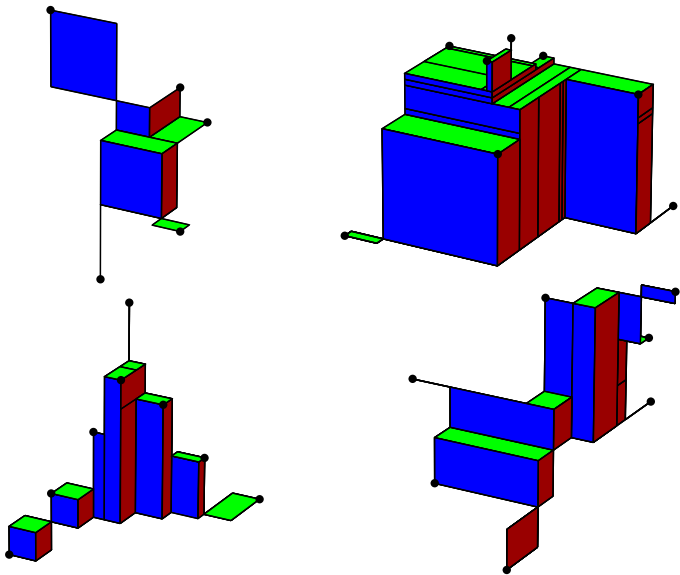

 $\mathcal{P}_{d_1}(T)$

 $\mathcal{P}_{d_2}(T)$

 $\mathcal{P}_{d_\infty}(T)$

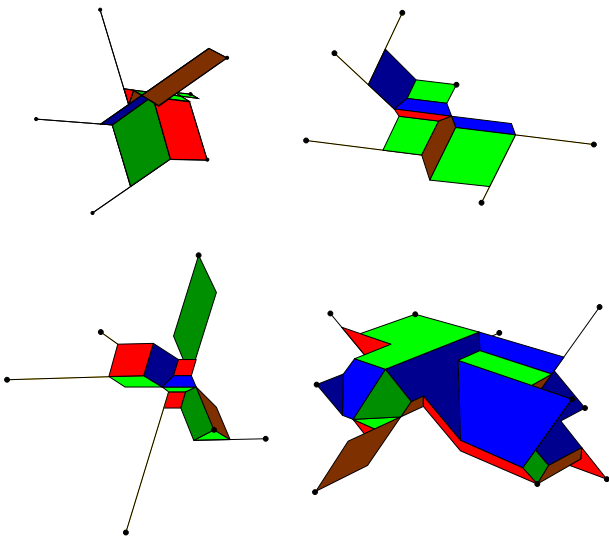
$\mathcal{P}_{d_1}(T)$ for 4 terminals



$\mathcal{P}_{d_1}^0(T)$ for 4 terminals



Pareto envelopes in ℓ_1 

Pareto envelopes in l_∞ 

Known results (1)

Applications : Pareto envelopes (known in literature as **sets of efficient points**) are used to reduce the search space in some optimization problems. They host all or at least one optimal solution(s) of problems like the **weighted median** problem, the NP-hard **multifacility location** problem, **Steiner tree** problem, as well as the **minimum Manhattan network** problem.

Thisse, Ward, and Wendell (1984) : The equality $\mathcal{P}_d(T) = \text{conv}(T)$ holds for all distances d defined by round norms.

Wendell and Hurter (1973) : The strict Pareto envelope $\mathcal{P}_d^0(T)$ is included in $\text{conv}(T)$ for all norms.

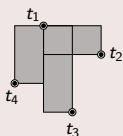
Durier, Michelot (1985,1986) introduced **elementary convex sets** as closed convex sets C of \mathbb{R}^m which are intersections of cones generated by facets of the unit ball and centered at terminals. They proved that for any polyhedral norm, the Pareto envelope is a connected union of such cells.



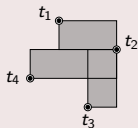
Known results (2)

Theorem (Chalmet, Francis, Kolen, 1981) : For any set of terminals T in (\mathbb{R}^2, d_1) :

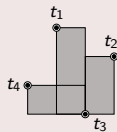
$$\mathcal{P}_{d_1}(T) = \bigcap_{i=1}^n (\bigcup_{j=1}^n I_{d_1}(t_i, t_j))$$



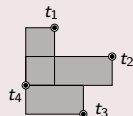
$$\bigcup_{j=1}^4 I_{d_1}(t_1, t_j)$$



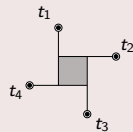
$$\bigcup_{j=1}^4 I_{d_1}(t_2, t_j)$$



$$\bigcup_{j=1}^4 I_{d_1}(t_3, t_j)$$



$$\bigcup_{j=1}^4 I_{d_1}(t_4, t_j)$$



$$\gamma_{d_1}(T)$$

Due to isometry between the ℓ_1 -plane and ℓ_∞ -plane, the same characterization holds for $\mathcal{P}_{d_\infty}(T)$.

This result is used to design an optimal $O(n \log n)$ time algorithm.

An $O(n^2)$ time algorithm proposed earlier by Wendell, Hurter, Lowe (1973).

Characterization of $\mathcal{P}_{d_1}(T)$

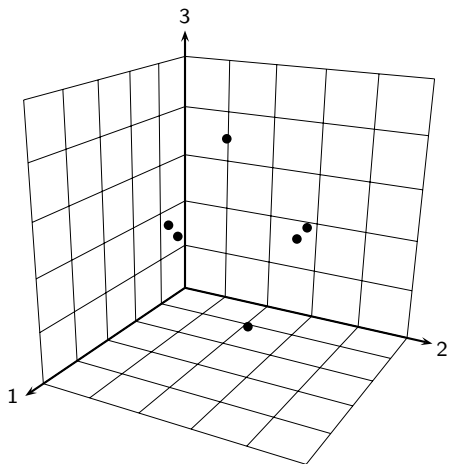
Theorem 1 : $\mathcal{P}_{d_1}(T) = \mathcal{M}(T) = \mathcal{I}(T)$ for any $T \subset \mathbb{R}^3$.

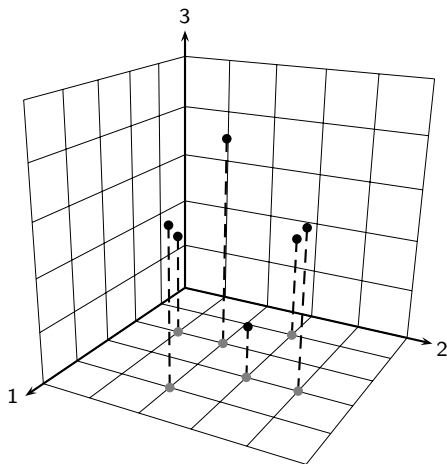
The set $\mathcal{M}(T)$ is a cubical complex which is the geometric realization of the **median closure** $\text{med}(T)$ of T .

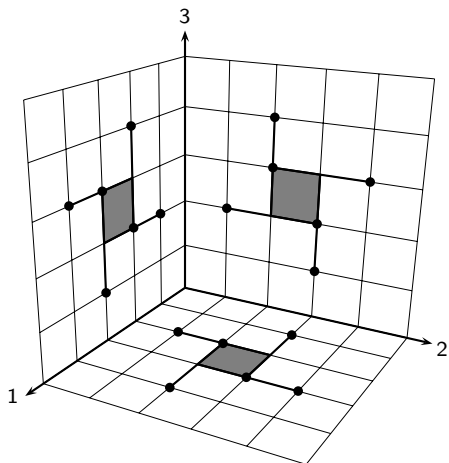
The set $\mathcal{I}(T)$ is the **intersection of three "cylindrical" sets** $\mathcal{I}^1(T), \mathcal{I}^2(T)$, and $\mathcal{I}^3(T)$ defined in the following way. For $i \in \{1, 2, 3\}$, denote by T^i the orthogonal projection of T on the hyperplane $H^i = \{x \in \mathbb{R}^3 : x^i = 0\}$ and let $\mathcal{P}_{d_1}(T^i)$ be the Pareto envelope of the set T^i in H^i . Then $\mathcal{I}^i(T)$ is the Cartesian product of $\mathcal{P}_{d_1}(T^i)$ with the line l^i orthogonal to the plane H^i .

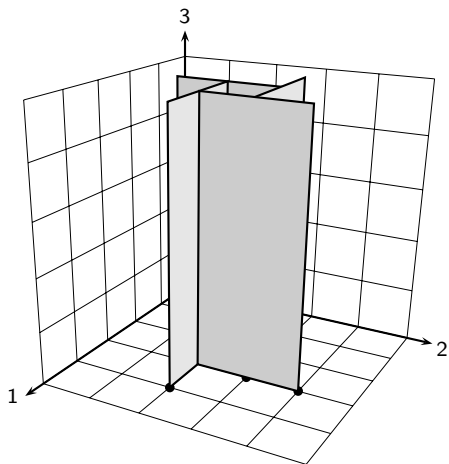
Remark : Unfortunately, this characterization of $\mathcal{P}_{d_1}(T)$ does not longer hold in higher dimensions.

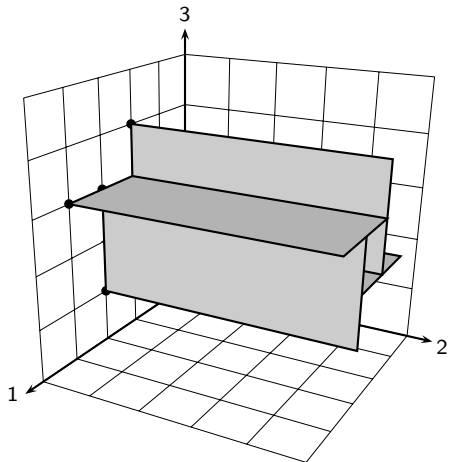


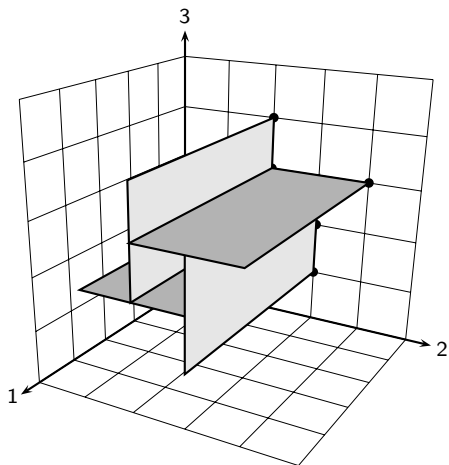
Definition of $\mathcal{I}(T)$ 

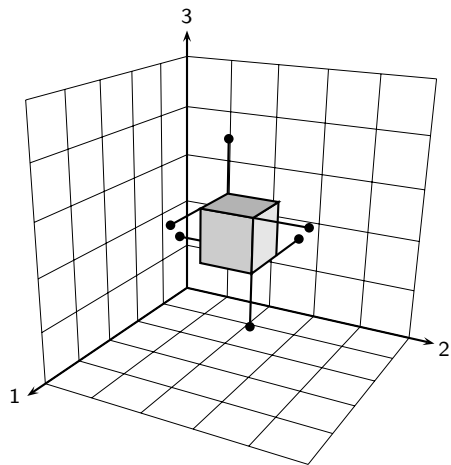
Definition of $\mathcal{I}(T)$ 

Definition of $\mathcal{I}(T)$ 

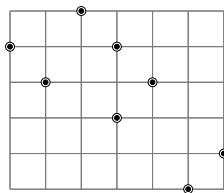
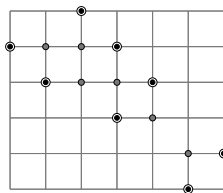
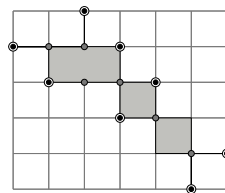
Definition of $\mathcal{I}(T)$ 

Definition of $\mathcal{I}(T)$ 

Definition of $\mathcal{I}(T)$ 

Definition of $\mathcal{I}(T)$  $\mathcal{P}_{d_1}(T)$

Definition of $\text{med}(T)$ and $\mathcal{M}(T)$

 T  $\text{med}(T)$  $\mathcal{M}(T)$

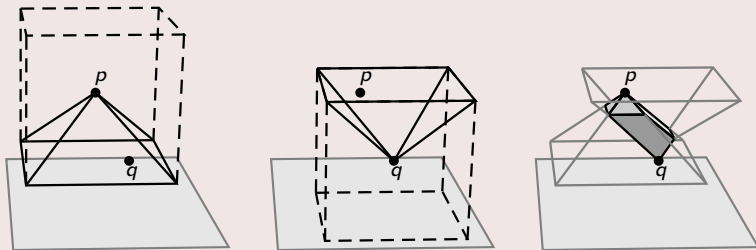
The **median** of three points $u, v, w \in \mathbb{R}^m$ is a (unique) point $m(u, v, w) = m \in \mathbb{R}^m$ such that m^i is the median value of u^i, v^i, w^i for all $i \in \{1, \dots, m\}$. A subset $S \subset \mathbb{R}^m$ is called **median stable** iff $m(u, v, w) \in S$ for each triplet $u, v, w \in S$.

The **median closure** $\text{med}(T)$ of T is the smallest median stable set $\text{med}(T)$ containing T . Roughly speaking, $\mathcal{M}(T)$ is obtained by replacing the discrete cubes of $\text{med}(T)$ by solid boxes.

Characterization of $\mathcal{P}_{d_\infty}(T)$

Theorem 2 : $\mathcal{P}_{d_\infty}(T) = \Upsilon_{d_\infty}(T)$ for any $T \subset \mathbb{R}^m$, where

$$\Upsilon_{d_\infty}(T) = \bigcap_{i=1}^n (\bigcup_{j=1}^n I_{d_\infty}(t_i, t_j)).$$



Construction of the interval

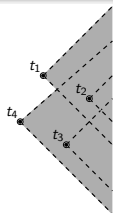
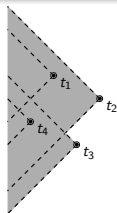
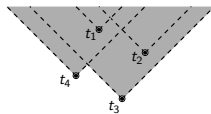
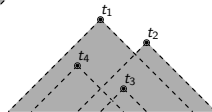
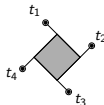
$$I_{d_\infty}(p, q) = \{r \in \mathbb{R}^m : d_\infty(p, q) = d_\infty(p, r) + d_\infty(r, q)\}.$$

Remark : Theorem 2 shows that the result of Chalmet, Francis, Kolen (1981) is characteristic for ℓ_∞ distances and not for ℓ_1 distances.

Characterization of $\mathcal{P}_{d_\infty}^0(T)$

Theorem 3 : $\mathcal{P}_{d_\infty}^0(T) = \mathcal{U}(T) := \bigcap_{i=1}^m (\mathcal{U}_+^i(T) \cap \mathcal{U}_-^i(T))$ for any $T \subset \mathbb{R}^m$.

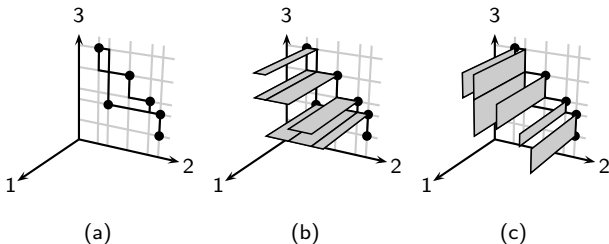
$\mathcal{U}_+^i(T) = \bigcup_{j=1}^n C_+^i(t_j)$ and $\mathcal{U}_-^i(T) = \bigcup_{j=1}^n C_-^i(t_j)$, where $C_+^i(p)$ and $C_-^i(p)$ are the cones having p as apex and induced by the two opposite facets of the unit cube centered at p .


 $\mathcal{U}_+^1(T)$

 $\mathcal{U}_-^1(T)$

 $\mathcal{U}_+^2(T)$

 $\mathcal{U}_-^2(T)$


$$\mathcal{U}(T) = \mathcal{U}_+^1(T) \cap \mathcal{U}_-^1(T) \cap \mathcal{U}_+^2(T) \cap \mathcal{U}_-^2(T)$$

Construction of $\mathcal{P}_{d_1}(T)$ in \mathbb{R}^3 **Algorithm ParetoR3L1 :**

1. Compute $\mathcal{P}_{d_1}(T^1)$, $\mathcal{P}_{d_1}(T^2)$, and $\mathcal{P}_{d_1}(T^3)$ using the algorithm of Chalmet et al. and preprocess it to obtain a **cut-strip representation**.
2. Construct the faces of the sets $\mathcal{I}^1(T)$, $\mathcal{I}^2(T)$, $\mathcal{I}^3(T)$.
3. Restrict each face of $\mathcal{I}^1(T)$, $\mathcal{I}^2(T)$, $\mathcal{I}^3(T)$ to $\mathcal{P}_{d_1}(T)$.

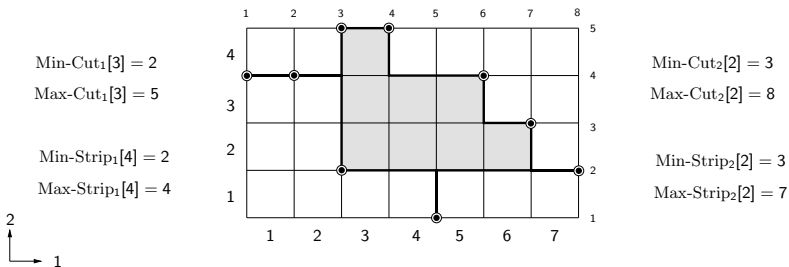


First two phases of ParetoR3L1

Cut-strip representation of $\mathcal{P}_{d_1}(T^i)$

A **cut** is the intersection of $\mathcal{P}_{d_1}(T^i)$ with a horizontal or vertical line passing via a terminal. A **horizontal strip** is the portion of $\mathcal{P}_{d_1}(T^i)$ comprised between two consecutive horizontal lines passing via terminals of T^i . The boundary of $\mathcal{P}_{d_1}(T^i)$ is subdivided into a linear number of **boundary segments** which are sides of strips of \mathcal{P} .

For each horizontal cut $c = [a, b]$, return the **ranks** of the abscises of a and b in the sorted list of abscises of the terminals. For each horizontal strip, return the **ranks** of the abscises of its vertical sides. For each vertical boundary segment s , return the **rank** of the horizontal strip defining it and of the the abscise of its end-vertices. The resulting arrays (of size $O(n)$) are called the **cut-strip representation** of $\mathcal{P}_{d_1}(T^i)$.

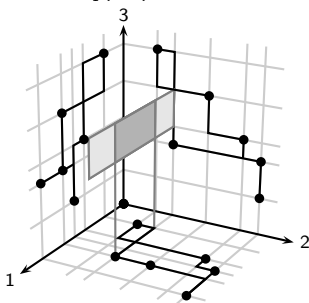


Phase 3 of ParetoR3L1

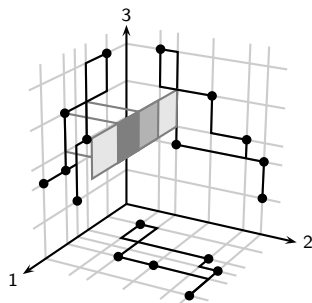
Let f be a face of $\mathcal{I}^1(T)$, i.e., $f = l^1 \times s$, where s is the vertical side s of a horizontal strip of $\mathcal{P}_{d_1}(T^1)$. First we compute $f' = f \cap \mathcal{I}^3(T)$ and then the rectangle $f'' = f' \cap \mathcal{I}^2(T)$, the final “contribution” of f to $\mathcal{I}(T)$.

Computation of f' : Find the cut of $\mathcal{P}_{d_1}(T^3)$ contained in the projection of f to the plane H^3 . The minimum and the maximum ranks of this cut defines the rectangle $f' = f \cap \mathcal{I}^3(T)$.

Computation of $f' \cap \mathcal{I}^2(T)$: Find the horizontal strip containing the projection of f' on H^2 . Its rank coincides with the rank of the horizontal strip of $\mathcal{P}_{d_1}(T^1)$ defining the boundary segment s . Using the minimum and the maximum ranks of this strip of $\mathcal{P}_{d_1}(T^2)$ and the coordinates of f' , compute f'' .



(a)



(b)

The complexity of $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 (1)

Theorem 4 : The boundary of the strict Pareto envelope $\mathcal{P}_{d_\infty}^0(T) = \bigcap_{i=1}^3 (\mathcal{U}_+^i(T) \cap \mathcal{U}_-^i(T))$ in \mathbb{R}^3 has linear complexity $\kappa(\mathcal{P}_{d_\infty}^0(T))$.

Sketch of the proof :

\mathcal{F} : the set of all faces of the cones from the families $\mathcal{U}_+^i(T)$ and $\mathcal{U}_-^i(T)$.

$\partial\mathcal{U}_+^i(T), \partial\mathcal{U}_-^i(T)$: the boundaries of the unions of cones $\mathcal{U}_+^i(T)$ and $\mathcal{U}_-^i(T)$, $i = 1, 2, 3$.

The 6 surfaces $\partial\mathcal{U}_-^i(T), \partial\mathcal{U}_+^i(T), i = 1, 2, 3$ subdivide \mathbb{R}^3 into cells which defines a **cell complex** $\Gamma_\infty^0(T)$.

By Theorem 3, $\mathcal{P}_{d_\infty}^0(T)$ is a subcomplex of $\Gamma_\infty^0(T)$. A cell C of $\Gamma_\infty^0(T)$ belongs to $\mathcal{P}_{d_\infty}^0(T)$ iff an interior point of C does.

We show that the complexity of $\Gamma_\infty^0(T)$ is linear.



The complexity of $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 (2)

For a face $F \in \mathcal{F}$, $Q_-^i(F)$ and $Q_+^i(F)$ are the intersections of F with $\partial\mathcal{U}_+^i(T)$ and $\partial\mathcal{U}_-^i(T)$, $i = 1, 2, 3$. These polygonal lines induce a subdivision of F into 2-dimensional cells, which are the 2-dimensional faces of $\Gamma_\infty^0(T)$.

The vertices and the pairwise intersections of $Q_-^i(F)$ and $Q_+^i(F)$, $i = 1, 2, 3$, define the vertices of $\Gamma_\infty^0(T)$.

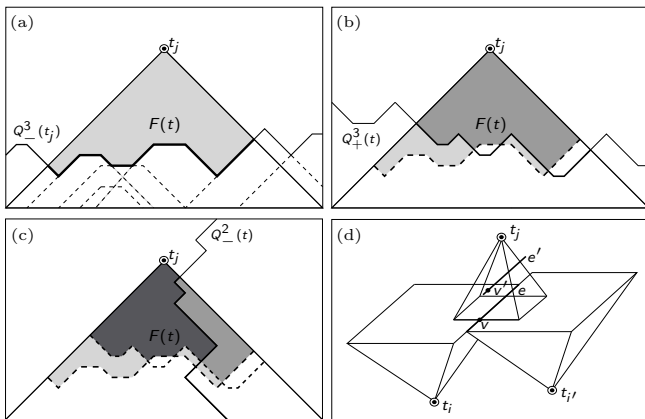
The subdivision of the polygonal chains $Q_-^i(F)$ and $Q_+^i(F)$ define the edges of $\Gamma_\infty^0(T)$.

$\mathcal{P}_{d_\infty}^0(T)$ and any cell of $\Gamma_\infty^0(T)$ are l^i -convex, where l^1, l^2, l^3, l^4 are the four lines defined by the opposite corners of the unit ball (cube) centered at the origin.

Every edge e of $\Gamma_\infty^0(T)$ belongs to a line $l(e)$ which is either parallel to one of the lines l^1, l^2, l^3, l^4 or is parallel to one of the coordinate lines m^1, m^2, m^3 .



The complexity of $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 (3)



The complexity of $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 (4)

Claim 1 : $\partial\mathcal{U}_-(T)$ and $\partial\mathcal{U}_+^i(T)$ contain $O(n)$ vertices.

Claim 2 : $\partial\mathcal{U}_-(T) \cap \partial\mathcal{U}_+^i(T)$ contains $O(n)$ vertices.

Claim 3 : Each of the sets $\partial\mathcal{U}_-(T) \cap \partial\mathcal{U}_+^i(T)$, $\partial\mathcal{U}_-^i(T) \cap \partial\mathcal{U}_+^j(T)$, and $\partial\mathcal{U}_+^i(T) \cap \partial\mathcal{U}_+^j(T)$ contain $O(n)$ vertices.

Claim 1 follows from the result of **Schwartz and Sharir (1990)** about linearity of lower envelopes of certain bivariate functions (it can be also proved directly). Claims 2 and 3 are proved using a **charging argument**.

By Claims 1,2, and 3, the total number of vertices on all chains is linear. For each $F \in \mathcal{F}$, the 6 polygonal chains $Q_-^i(F)$, $Q_+^i(F)$, $i = 1, 2, 3$ are l^i -convex, $i = 1, \dots, 4$, hence the total number of the intersection points of any two from the 6 chains of F is linear in the size of respective chains. Since every vertex of $\Gamma_\infty^0(T)$ is obtained in this way, the proof of Theorem 4 is finished.

Construction of $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3

Algorithm ParetoR3LINF

1. Compute the surfaces $\partial\mathcal{U}_-(T)$ and $\partial\mathcal{U}_+(T)$ of $\mathcal{U}_-(T)$ and $\mathcal{U}_+(T)$, $i = 1, 2, 3$ as lower envelopes of bivariate functions of **Schwartz and Sharir (1990)**. Preprocess each of 6 surfaces to support ray shootings in 14 directions defined by the lines l^1, l^2, l^3, l^4 and m^1, m^2, m^3 in fixed-oriented polyhedra as described by **M. de Berg (1993)**.
2. For each face F , compute the 6 polygonal chains $Q_-^i(F), Q_+^i(F)$, $i = 1, 2, 3$, by performing repetitive ray shootings in the directions belonging to the face F .
3. Compute the subdivision of any $F \in \mathcal{F}$ into faces of $\Gamma_\infty^0(T)$ induced by the chains $Q_-^i(F), Q_+^i(F)$, $i = 1, 2, 3$. The intersection points are found by sweeping any pair of these chains.
4. Compute $\Gamma_\infty^0(T)$ by merging the incidence lists of the vertices of the subdivisions obtained in Phase 3.
5. Compute the cells C of $\Gamma_\infty^0(T)$ belonging to $\mathcal{P}_{d_\infty}^0(T)$. Pick any interior point p of C and add C to $\mathcal{P}_{d_\infty}^0(T)$ iff $p \in \bigcap_{i=1}^3 (\mathcal{U}_+^i(T) \cap \mathcal{U}_-^i(T)) = \mathcal{P}_{d_\infty}^0(T)$. For this, compute the intersection of each $\partial\mathcal{U}_+^i(T)$ and $\partial\mathcal{U}_-^i(T)$ with the line passing via p and parallel to l^1 . To find this intersection, perform two ray shootings from p in the opposite directions defined by l^1 .



Main results

Theorem 1 : $\mathcal{P}_{d_1}(T) = \mathcal{M}(T) = \mathcal{I}(T)$ for any $T \subset \mathbb{R}^3$.

Theorem 2 : $\mathcal{P}_{d_\infty}(T) = \Upsilon_{d_\infty}(T)$ for any $T \subset \mathbb{R}^m$.

Theorem 3 : $\mathcal{P}_{d_\infty}^0(T) = \mathcal{U}(T)$ for any $T \subset \mathbb{R}^m$.

Theorem 4 : For any n -point subset T of \mathbb{R}^3 , $\mathcal{P}_{d_1}(T)$ and $\mathcal{P}_{d_\infty}^0(T)$ have complexity $O(n)$.

Theorem 5 : For any n -point subset T of \mathbb{R}^3 , $\mathcal{P}_{d_1}(T)$ can be constructed in optimal $O(n \log n)$ time and $\mathcal{P}_{d_1}^0(T)$ can be constructed in $O(n \log^2 n)$ time.



Open questions

Question 1 : Characterize $\mathcal{P}_{d_1}(T)$ in dimensions $m > 3$.

Question 2 : Design an $O(n \log n)$ algorithm for $\mathcal{P}_{d_\infty}(T)$ and $\mathcal{P}_{d_\infty}^0(T)$ in \mathbb{R}^3 .

Question 3 : Design efficient algorithms for $\mathcal{P}_{d_1}(T)$ and $\mathcal{P}_{d_\infty}(T)$ in dimensions $m > 3$. What is the complexity of these envelopes in higher dimensions?

