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Discrete Mathematics 183 (1998) 47–60

DISCRETE MATHEMATICS

A note on r-dominating cliques[1]

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Received 12 June 1995; received in revised form 15 January 1997; accepted 24 February 1997

Abstract

Let M be a finite subset of vertices of a connected graph G and assume that every vertex $v \in M$ has a dominating radius $r(v) \in \mathbb{N} \cup \{0\}$. A complete subgraph C is an *r-dominating clique* of M if every vertex $v \in M$ is at distance at most $r(v)$ from C. Even for $r(v) \equiv 1$ the problem whether or not a given graph has an r-dominating clique is NP-complete. Evidently, if M admits an r-dominating clique then $d(u,v) \leqslant r(u)+r(v)+1$ for any $u,v \in M$. We characterize the graphs G for which this condition guarantees the existence of r-dominating cliques not only in G but also in all isometric subgraphs of G comprising M. These are the graphs which do not contain the house, the 3-deltoid, or any n-cycle with $n \geqslant 5$ as an isometric subgraph.

1. Introduction and the result

In recent years several kinds of domination problems in graphs have been investigated. In all of them it is necessary to find a subgraph Q with a prescribed structure which dominates all or certain vertices of a graph G. Recall that Q dominates a subset of vertices M if every vertex of M outside Q is adjacent to some vertex of Q. In a more general situation each vertex $v \in M$ is endowed with a *dominating radius* $r(v)$, which is a non-negative integer. Then Q *r-dominates* the set M if every vertex $v \in M$ is at distance at most $r(v)$ from some vertex of Q. Apart from the generic domination problem, when Q can be an arbitrary subgraph of G, the special cases when Q is selected among connected subgraphs, cycles (not necessarily induced), cliques, or edge-free subgraphs present a considerable interest. All these problems are NP-complete even for very special classes of graphs; for a bibliography on domination cf. [17].

The present work addresses the following r-dominating clique problem: find a clique C (if it exists) such that every vertex $v \in M$ is at distance at most $r(v)$ from some

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[1] Research supported by the Alexander von Humboldt Stiftung.

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PII S0012-365X(97)00076-9

vertex of C. If such a clique C exists, we will say that C is an *r-dominating clique* of M. As was shown in [7] even for weakly triangulated graphs the problem to decide whether or not a dominating clique ($r(v)\equiv 1$) exists is NP-complete. Evidently, if M has an r-dominating clique then the distance $d(u,v)$ between any vertices $u,v\in M$ is at most $r(u)+r(v)+1$. As was shown in [11–13] the converse is true for all chordal graphs, Helly graphs, distance-hereditary graphs and house-hole-domino-free (HHD-free) graphs; for the case of chordal graphs and $r(v)\equiv 1$ see [20,21].

The object of this note is to characterize in terms of forbidden isometric subgraphs the class of graphs G with the following hereditary property:

For every finite subset of vertices M and an arbitrary collection of r-dominating radii of vertices of M every isometric subgraph containing M has an r-dominating clique of M if and only if $d(u,v)\leqslant r(u)+r(v)+1$ for any $u,v\in M$.

From the proof of this result follows that for such finite graphs an r-dominating clique (if it exists) can be constructed in polynomial time. This class of graphs comprises the important classes of chordal bipartite graphs [15], hereditary modular [1] and hereditary pseudo-modular [3] graphs, chordal graphs [8,10], HHD-graphs [19], house-free weakly triangulated graphs [16], and 3-deltoid-free bridged graphs [14,24].

The graphs in this note are connected but not necessarily finite. The *distance* $d(u,v)$ between two vertices u and v of a graph G is the length of a shortest path between u and v. The set of all vertices w on shortest paths between u and v is called the *interval* $I(u,v)$ between u and v, i.e.,

$$I(u,v)=\{w\colon\ d(u,v)=d(u,w)+d(w,v)\}.$$

An induced subgraph H of a graph G is *isometric* if the distance between any pair of vertices in H is the same as that in G.

A graph G is *weakly modular* [2,5,9] if its shortest-path metric $d=d_G$ satisfies the following two conditions (for an illustration see Fig. 1):

Triangle condition: for any three vertices u,v,w with

$$1=d(v,w)\ <\ d(u,v)=d(u,w)=k$$

there exists a common neighbour x of v and w such that $d(u,x)=k-1$;

Quadrangle condition: for any four vertices u,v,w,z with

$$d(v,z)=d(w,z)=1 \quad\text{and}\quad k=d(u,v)=d(u,w)=d(u,z)-1,$$

there exists a common neighbour x of v and w such that $d(u,x)=k-1$.

Then a *hereditary weakly modular* graph G is a weakly modular graph such that every isometric subgraph of G is also weakly modular.

We are now in a position to formulate the main result.

Theorem. *For a graph $G=(V,E)$ the following conditions are equivalent*:

(1) *every finite subset $M\subseteq V$ with*

$$d(u,v)\leqslant r(u)+r(v)+1$$

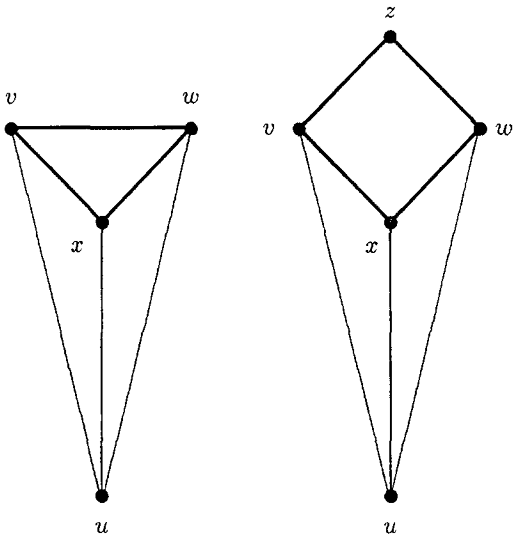


Fig. 1. The triangle and the quadrangle conditions.

for all $u,v\in M$ *has an* r*-dominating clique in every isometric subgraph* H *comprising* M;

(2) G *does not contain the house, the* 3*-deltoid, or any* n*-cycle with* $n\geqslant 5$ *as an isometric subgraph.*

Now, we consider the hereditary property with respect to the family of all connected induced subgraphs of a given graph G. A graph G obeying this property does not contain houses and cycles of length at least 5 as induced subgraphs (see Fig. 2). Therefore G is a house-free weakly triangulated graph (recall, that a graph is called *weakly triangulated* [16], if it does not contain a cycle C_n, $n\geqslant 5$, or its complement as an induced subgraph). Conversely, if G is a house-free weakly triangulated graph, then G and any of its induced subgraph H do not contain the house, the 3-deltoid, or any cycle $n\geqslant 5$ as an induced (and, therefore, isometric) subgraph. By our theorem H has an r-dominating clique for any finite subset M, such that $d_H(u,v)\leqslant r(u)+r(v)+1$ for all $u,v\in M$. We formulate this result in the following form.

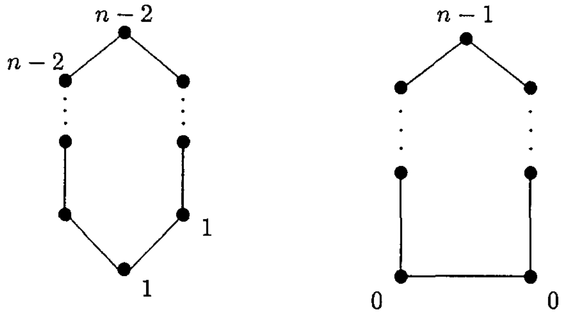
Corollary 1. *For a graph* $G=(V,E)$ *the following conditions are equivalent*:

(1) *for any connected induced subgraph* H *containing a finite subset* $M\subseteq V$, M *has an* r*-dominating clique in* H *if and only if*

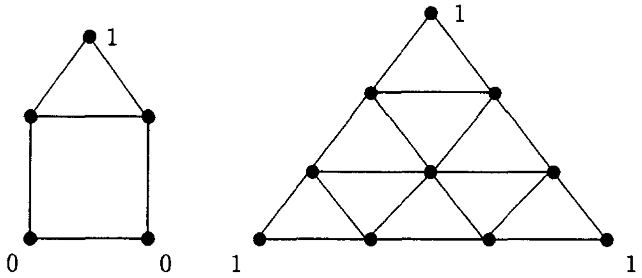
$$d_H(u,v)\leqslant r(u)+r(v)+1$$

for all $u,v\in M$;

(2) G *is a house-free weakly triangulated graph.*



even cycle C_{2n} odd cycle C_{2n+1}



house 3–deltoid

Fig. 2. The forbidden graphs.

Let $d(G)$ and $r(G)$ be the *diameter* and the *radius* of a graph G. From the condition (1) of the theorem we immediately obtain a sharp relationship between radii and diameters of such graphs. The result generalizes similar inequalities for trees, chordal graphs, distance-hereditary graphs and house-hole-domino-free graphs. The proof follows that one presented in [13] for HHD-free graphs.

Corollary 2. *If a graph* $G=(V,E)$ *does not contain the house, the* 3*-deltoid, or any* n*-cycle with* $n\geqslant 5$ *as an isometric subgraph, then*

$$d(G)\geqslant 2r(G)-2.$$

Proof. For any vertex $v\in V$ define $r(v)=r(G)-2$. If $d(G)\leqslant 2r(G)-3$, then for any vertices $u,v\in V$

$$d(u,v)\leqslant 2r(G)-3=r(u)+r(v)+1.$$

According to the theorem G has an r-dominating clique C. For any vertices $x\in C$ and $v\in V$ we obtain $d(x,v)\leqslant r(v)+1=r(G)-1$, contrary to the definition of the radius of G. □

2. Preliminary results

In this section we present some properties of weakly modular graphs. We begin with a characterization of weakly modular graphs and of hereditary weakly modular graphs.

Theorem A (Chepoi [9, Theorem 2]). *For a graph G the following conditions are equivalent*:

(1) *G is weakly modular*;

(2) *for any three vertices u,v,w such that the intervals $I(u,v)$, $I(v,w)$ and $I(w,u)$ pairwise intersect only in the common end vertices, all vertices of the interval $I(u,v)$ are equidistant from w.*

As is noted in [2] the same characterization of weakly modular graphs can be derived from Lemma 1 of [5].

Theorem B (Chepoi [9, Theorem 5]). *For a graph G the following conditions are equivalent*:

(1) *G is hereditary weakly modular*;

(2) *G is weakly modular and does not contain the house, the 5-cycle or the 6-cycle as an isometric subgraph*;

(3) *G does not contain the house or any n-cycle with $n \geqslant 5$ as an isometric subgraph.*

The following simple fact, extending a well-known property of the chordal graphs, is often useful.

Lemma 1. *Let G be a house-free weakly modular graph. If all vertices of a clique C are at the same distance k from a vertex x then there is a common neighbour x^* of all vertices of C such that $d(x,x^*)=k-1$.*

Proof. Let x^* be a vertex at distance $k-1$ from x which has maximum number of neighbours in C. Suppose that x^* is not adjacent to some vertex $v \in C$. Pick an arbitrary neighbour u of x^* in C. By the triangle condition there is a common neighbour x' of u and v, such that $d(x,x')=k-1$. Consequently, applying the quadrangle condition to the vertices $x',x^* \in I(u,x)$, we will get a common neighbor y of x' and x^*, which is one step closer to x. Since G is house-free, necessarily x^* and x' will be adjacent. From the choice of the vertex x^* we conclude that x' is not adjacent to at least one neighbour w of x^* in C. Then, however, the vertices y, x^*, x', w, v induce a house. □

Lemma 2. *If Γ is an induced 6-cycle of a hereditary weakly modular graph G, then there is a vertex adjacent to all vertices of Γ. In particular, $d(u,v) \leqslant 2$ for any $u,v \in \Gamma$.*

Proof. As Γ cannot be isometric, necessarily two opposite vertices of Γ, say u and v, are at distance two. Let w be a neighbour of v in Γ. By Theorem B (1)$\Rightarrow$(3) Γ

cannot be isometric. Hence two opposite vertices in Γ have a common neighbour x. By Theorem B $(1)\Rightarrow(2)$ G does not contain a 5-cycle or a house. Consequently x must be adjacent to the remaining vertices in Γ. □

Lemma 3. *If* $\Gamma=(v_1,v_2,v_3,v_4,v_5,v_6,v_1)$ *is a 6-cycle of a hereditary weakly modular graph* G *and* $d(v_1,v_4)=3$, *then* Γ *contains at least one of the diagonals* v_2v_5 *or* v_3v_6.

Proof. Suppose by way of contradiction that $v_2v_5, v_3v_6\notin E$. By Lemma 2 the cycle Γ cannot be induced. Therefore v_2,v_6 or v_3,v_5 must be adjacent. But then we will get either an induced 5-cycle or a house, which is impossible. □

3. Proof of the theorem

$(1)\Rightarrow(2)$: Each of the forbidden graphs violates the condition (1), as indicated in Fig. 2. In each case the set M which does not admit an r-dominating clique consists of the vertices with labels; each label represents the dominating radius of a respective vertex.

$(2)\Rightarrow(1)$: Let M be a finite subset of G such that $d(u,v)\leqslant r(u)+r(v)+1$ for any $u,v\in M$. We assert that M has an r-dominating clique in every isometric subgraph of G which contains M. Since every such subgraph satisfies the condition (2), it suffices to establish our assertion for the graph G only. By Theorem B G is a hereditary weakly modular graph.

We begin by sketching how to find an r-dominating clique of M. Given a current clique C which already r-dominates some vertices of M (initially we can suppose that C is an arbitrary vertex of M or even an arbitrary clique C_0 of G), we construct a new clique C^* which r-dominates the same vertices of M as C and is one step closer to some still non-dominated by C vertex v of M. Put $C:=C^*$. If v is not dominated by the new current clique, then repeat the procedure for v. Otherwise, start the same procedure with a new non-dominated by C vertex of M. Evidently, after at most $\sum_{v\in M} d(C_0,v)$ iterations we will get a clique which r-dominates the whole set M. Below we explain in detail how to transform C into a new current clique C^*.

Let C be the current clique. Henceforth, we denote by $C(u)$ the *metric projection* of a vertex u in C, i.e.

$$C(u)=\{x\in C:\ d(u,x)=d(u,C)\},$$

where $d(u,C)=\min\{d(u,y):\ y\in C\}$. According to Lemma 1 there is a common neighbour u^* of all vertices of $C(u)$ such that $d(u,u^*)=d(u,C)-1$. We will say that u^* is a *gate* of u in the clique C.

Let us suppose that C r-dominates the vertices of some proper subset M^* of M. We fix an arbitrary vertex $v\in M-M^*$. First suppose that every $u\in M^*$ is at distance at most $r(u)$ from the clique $C(v)$. Then define a new clique $C^*=C(v)\cup\{v^*\}$, where v^* is an arbitrary gate of v in the clique C. Clearly, C^* r-dominates all vertices of

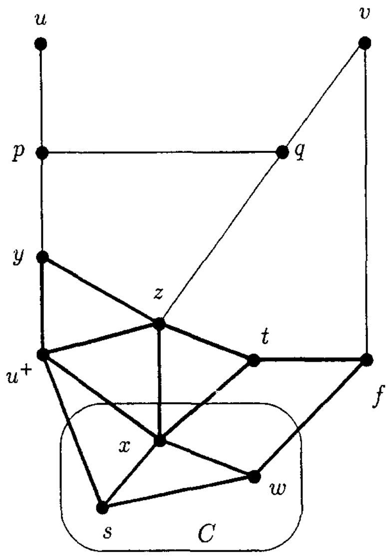


Fig. 3.

M^*. Since, C^* is one step closer to v than C, we can replace C by C^*. Hence, further we may assume that there is at least one vertex $u \in M^*$ at distance $r(u)+1$ from $C(v)$. This yields $d(u,C)=r(u)$ and $C(u) \cap C(v)=\emptyset$. Denote the collection of all such vertices u by $M^+(v)$. Let also $d(v,C)=k$. In Figs. 3–6 thick lines indicate edges while thin lines indicate shortest paths.

Claim 1. *Every vertex $u \in M^+(v)$ has a gate in C whose distance to v is at most $k+1$.*

Proof. Take an arbitrary gate u^+ of u in C and suppose by way of contradiction that $d(u^+,v) > k+1$. This implies $d(u^+,v)=k+2$, since C is a clique. Then any vertex of $C(u)$ belongs to the interval $I(u^+,v)$. First suppose that there is a neighbour u' of u^+ in the intersection $I(u^+,u) \cap I(u^+,v)$. Pick $x \in C(u)$. As $u',x \in I(u^+,v)$, by the quadrangle condition applied to the vertices u',u^+,x,v we can find a common neighbour u^* of u' and x with $d(u^*,v)=k$. We assert that u^* is adjacent to any other vertex $s \in C(u)$. Indeed, if u^* and s were non-adjacent, then the vertices s,x,u^+,u' and u^* would induce a house, which is impossible (the choice of u' yields that s and u' are not adjacent). Thus u^* is adjacent to any $s \in C(u)$. Since u^* and u' are adjacent, we obtain that u^* and u' belong to a common shortest path between the vertices s and u. Hence u^* is a gate of u in C such that $d(v,u^*)=k$, and we are done.

Next suppose that $I(u^+,u) \cap I(u^+,v)=\{u^+\}$. Consider a farthest from u vertex p in the set $I(u,u^+) \cap I(u,v)$. Then $I(p,u^+) \cap I(p,v)=\{p\}$. Let also q be a furthest from v vertex in $I(v,u^+) \cap I(v,p)$. The choice of p and q implies that $I(p,u^+) \cap I(p,q)=\{p\}$ and $I(q,p) \cap I(q,u^+)=\{q\}$. Since $d(u,v) \leqslant r(u)+r(v)+1$, and $d(u,u^+)=r(u)-1$,

$d(u^+,v)=k+2\geqslant r(v)+3$, we obtain that $u^+\neq p$ and $u^+\neq q$. Note also that $p\neq q$, otherwise $p\in I(u^+,u)\cap I(u^+,v)$, contrary to our assumption. Pick an arbitrary neighbour y of u^+ in the interval $I(u^+,p)$. Theorem A entails that $d(q,y)=d(q,u^+)$. By the triangle condition apllied to vertices y,u^+ and q we can find a common neighbour z of y and u^+ which is one step closer to q. Let $x\in C(u)$. Since $z,x\in I(u^+,v)$, by the quadrangle condition applied to vertices z,x,u^+,v we can find a common neighbour $t\in I(z,v)\cap I(x,v)$ of the vertices z and x. The vertices y and t are non-adjacent, otherwise $y\in I(u^+,u)\cap I(u^+,v)$, contrary to our assumption. Then z will be adjacent to x, because the vertices y,z,u^+,t and x cannot induce a house.

We will prove that z is a gate of u in C. Suppose the contrary: then there exists a vertex $s\in C(u)$ which is non-adjacent to z. Note that $t\notin C(v)$, for otherwise $t\in C(v)\cap C(u)$. Take a vertex $w\in C(v)$. Since $t,w\in I(x,v)$, by the quadrangle condition there is a common neighbour f of t and w at distance $k-1$ from v; see Fig. 3. In the subgraph induced by the vertices s,x,w,t, and f the vertex t must be adjacent to at least one of the vertices s or w, otherwise we would get an induced house. If $ts\in E$, then the vertices y,z,u^+,s,t induce a house. So, suppose that t and w were adjacent. But then we get an induced 5-cycle (s,u^+,z,t,w,s). Thus, z is a gate of u with the desired property that $d(z,v)\leqslant k+1$. □

Therefore we can split the set $M^+(v)$ into subsets $M'(v)$ and $M''(v)$, where

$$M'(v)=\{u\in M^+(v)\text{: a gate of } u \text{ in } C \text{ closest to } v \text{ is at distance } k+1 \text{ from } v\},$$

$$M''(v)=\{u\in M^+(v)\text{: a gate of } u \text{ in } C \text{ closest to } v \text{ is at distance } k \text{ from } v\}.$$

In following Claims 2 and 3 we assume that $M''\neq\emptyset$.

Claim 2. *For any vertex $u\in M''(v)$ we have $C(u)=C-C(v)$. If u^* is a gate of u such that $d(u^*,v)=k$, then u^* is adjacent to a gate of v.*

Proof. Suppose the contrary: then there is a vertex $z\in C$ outside the sets $C(u)$ and $C(v)$. Take two vertices $x\in C(u)$ and $y\in C(v)$, and a gate u^* of u which is at distance k from v. By the quadrangle condition applied to u^*,x,y and v we can find a common neighbour t of the vertices y and u^*, which is one step closer to v. But now x,u^*,y,t and z induce a house, because $C(u)\cap C(v)=\emptyset$ and $z\notin C(u)\cup C(v)$. Let s be any vertex of $C(v)$. If $st\notin E$, then the vertices u^*,x,y,s and t induce a forbidden house. Thus t is a gate of v in C adjacent to u^*. □

Claim 3. *There is a gate of v which is adjacent to at least one gate of each vertex of $M''(v)$.*

Proof. Among the gates of the vertex v we select a gate v^* which is adjacent to gates of maximum number of vertices from $M''(v)$. Suppose by way of contradiction that there is a vertex $w \in M''(v)$ without a gate in C adjacent to v^*. By Claim 2 we can find two adjacent vertices w^* and v^+, which are gates in C of w and v, respectively. Then $v^*, v^+ \in I(y,v)$ for any $y \in C(v)$. By the quadrangle condition applied to vertices v^+, y, v^*, v there is a common neighbour p of v^* and v^+ at distance $k-2$ to v.

Now, consider an arbitrary vertex $u \in M''(v)$ such that some its gate u^* in C is adjacent to v^*. We assert that v^+ is adjacent to any such u^*. Indeed, by Claim 2 both u^* and w^* are adjacent to all vertices $x \in C - C(v)$. Consider the 6-cycle $(x, u^*, v^*, p, v^+, w^*, x)$. Since $d(x,p)=3$ and $v^*w^* \notin E$, by Lemma 3 we deduce that $u^*v^+ \in E$. This contradicts the choice of v^*. Hence, v^* must be adjacent to a gate of every vertex from $M''(v)$. □

For the rest of the proof, denote by R'' the collection of gates of the vertex v which fulfill the condition of Claim 3. We continue by establishing some properties of gates of the vertices from $M'(v)$.

Claim 4. *If $u \in M'(v)$ and u^* is a gate of u such that $d(u^*, v) = k+1$, then there is a gate v^* of v which fulfills the following conditions:*
(a) *$v^* \in I(u^*, v)$ and $d(u^*, v^*) = 2$;*
(b) *any common neighbour t of u^* and v^* is adjacent to all vertices of $C(u) \cup C(v)$.*

Proof. (a) Since $C(v) \cap C(u) = \emptyset$, all vertices of $C(u)$ are at distance $k+1$ from v. By Lemma 1 applied to the clique $C(u) \cup \{u^*\}$ there is a vertex s which is adjacent to u^* and all vertices of $C(u)$. Pick arbitrary vertices $x \in C(u)$ and $y \in C(v)$. Since $y, s \in I(x,v)$ and x is adjacent to s and y, by the quadrangle condition there is a common neighbour v^* of s and y with $d(v^*, v) = k-1$. The vertices s and y must be adjacent, otherwise u^*, s, v^*, y and x induce a house. We can assume that v^* is not adjacent to some vertex $z \in C(v)$, otherwise v^* is a gate of v which satisfies the condition (a). Let v^+ be an arbitrary gate of v. Since $v^+, v^* \in I(y,v)$, by the quadrangle condition there is a common neighbour p of v^+ and v^* with $d(p,v) = k-2$. Since $d(x,p) = 3$, by Lemma 3 we conclude that in the 6-cycle $(x, s, v^*, p, v^+, z, x)$ the vertices s and v^+ must be adjacent. Therefore v^+ is a gate of v which obeys (a).

(b) Let u^* and v^* be the gates of u and v, respectively, such that $d(u^*, v^*) = 2$, and let t be their common neighbour. Pick two arbitrary vertices $x \in C(u)$ and $y \in C(v)$. Consider the 5-cycle (x, u^*, t, v^*, y, x), which cannot be induced. Since $C(u) \cap C(v) = \emptyset$, necessarily t must be adjacent to both x and y. □

Henceforth, we shall use the following notations. For each $u \in M'(v)$ let $R(u)$ consists of all gates of v which are at distance 2 from some gate of u. Let also $Q(u)$ denote the collection of all vertices t, for which condition (b) of Claim 4 is fulfilled. From Claim 4 we know that both $R(u)$ and $Q(u)$ are non-empty.

Claim 5. *If $u, w \in M'(v)$ and $C(u) \cap C(w) \neq \emptyset$, then any two vertices $t \in Q(u) - Q(w)$ and $s \in Q(w) - Q(u)$ are adjacent.*

Proof. Suppose by way of contradiction that $ts \notin E$. Let $t \in I(u^*, v_1^*)$ and $s \in I(w^*, v_2^*)$, where $v_1^* \in R(u), v_2^* \in R(w)$, and u^* and w^* be some gates of u and w at distance 2 from v_1^* and v_2^*, respectively. Choose an arbitrary vertex $x \in C(u) \cap C(w)$. Note that $v_1^* \neq v_2^*$, otherwise we would get two houses induced by the vertices u^*, x, v_1^*, s, t and w^* (the initial condition implies $u^*s, w^*t \notin E$). Hence, we may assume that the vertices t, v_2^* and s, v_1^* were non-adjacent. As $v_1^*, v_2^* \in I(y, v)$ for $y \in C(v)$, by the quadrangle condition there exists a common neighbour p of v_1^* and v_2^* at distance $k-2$ to v. Since $d(x, p) = 3$, by Lemma 3 we conclude that in the 6-cycle $(x, t, v_1^*, v^*, v_2^*, s, x)$ either $sv_1^* \in E$ or $tv_2^* \in E$, which is impossible. This proves that the vertices t and s must be adjacent. □

Claim 6. *For any vertices* $u, w \in M'(v)$ *the sets* $R(u)$ *and* $R(w)$ *are comparable, i.e., either* $R(u) \subseteq R(w)$ *or* $R(w) \subseteq R(u)$.

Proof. Suppose the contrary, i.e., there exist two vertices $v_1^* \in R(u) - R(w)$ and $v_2^* \in R(w) - R(u)$. As $v_1^*, v_2^* \in I(y, v)$ for any $y \in C(v)$, by the quadrangle condition applied to vertices y, v_1^*, v_2^* and v there is a common neighbour v^* of v_1^* and v_2^* which is one step closer to v. Let u^* and w^* be some gates of u and w at distance 2 from v_1^* and v_2^*, respectively. Pick arbitrary vertices $t \in I(u^*, v_1^*) \cap Q(u)$ and $s \in I(w^*, v_2^*) \cap Q(w)$. From the choice of v_1^* and v_2^* we conclude that $t \notin Q(w)$ and $s \notin Q(u)$, and, in addition, $d(u^*, v_2^*) = d(w^*, v_2^*) = 3$. If $C(u) \cap C(w) \neq \emptyset$, by Claim 5 we will obtain that t and s are adjacent. Then, however, the vertices t, v_1^*, v^*, v_2^*, s induce a 5-cycle or a house. Thus $C(u) \cap C(w) = \emptyset$ and $ts \notin E$. Choose arbitrary vertices $x \in C(u), z \in C(w)$ and $y \in C(v)$. By Claim 4 t must be adjacent to x and y, while s must be adjacent to z and y; see Fig. 4. Since G is house-free, the vertices v_1^* and v_2^* are adjacent. Moreover, $tz, sx \notin E$, otherwise we will get a 5-cycle induced by t, v_1^*, v_2^*, s, and t or by s, v_2^*, v_1^*, t, and z. Note that $d(u^*, w^*) \geqslant 2$, otherwise, if $u^*w^* \in E$, the vertices y, z, x, w^*, u^* induce a house.

Now, suppose that $d(u^*, w^*) = 2$, i.e., there is a common neighbour q of u^* and w^*. In order to avoid forbidden subgraphs induced by u^*, x, z, w^* and q, the vertex q must be adjacent to both x and z. Then we get two 5-cycles (q, u^*, t, y, z, q) and (q, x, y, s, w^*, q), which cannot be induced. Since G is house-free and $u^*y, u^*z, w^*y, w^*x \notin E$, the vertex q will be adjacent to t, s and y. Now, if we consider the 5-cycle $(q, t, v_1^*, v_2^*, s, q)$, we obtain that q must be adjacent to both v_1^* and v_2^*. Then, however, we deduce that $q \in Q(u) \cap Q(w)$. Consequently, $v_1^*, v_2^* \in R(u) \cap R(w)$, contrary to our assumption.

Finally, suppose that $d(u^*, w^*) = 3$. Note that the vertices $u^*, x, z, w^*, s, y, t, v_1^*, v_2^*$ and v^* induce a 3-deltoid, which cannot be isometric. Since v^* is at distance 3 from each of the vertices u^*, x, z and w^*, and also $d(u^*, v_2^*) = d(w^*, v_1^*) = 3$, we can suppose that $d(u^*, s) = 2$. By Lemma 1 there is a common neighbour p of the vertices u^*, z, y and s. Since G is house-free and $tz, sx, st \notin E$, the vertex p must be adjacent to t and x (otherwise a house induced by s, y, p, t, u^* or by s, z, p, x, u^* occurs). In the obtained 5-cycle $(p, t, v_1^*, v_2^*, s, p)$ the vertex p must be adjacent to both v_1^* and v_2^*. Then,

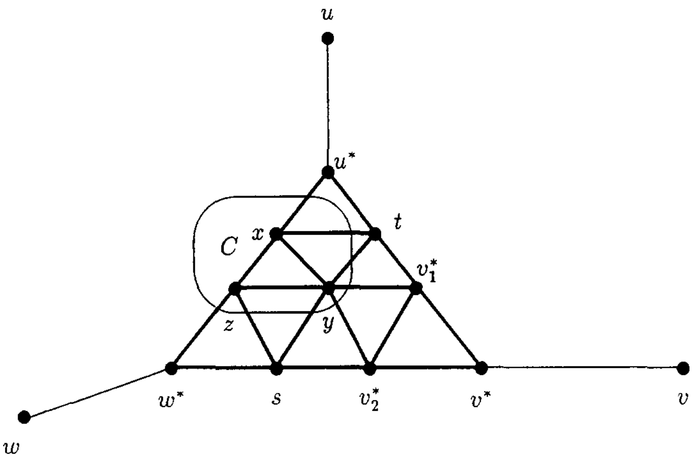


Fig. 4.

however, $d(u^*, v_2^*) = 2$, contrary to our assumption. This shows that the sets $R(u)$ and $R(w)$ are comparable. □

By virtue of Claim 6, there is at least one gate v^* of v which belongs to every set $R(u), u \in M'(v)$. Denote by R' the collection of such vertices v^*.

Claim 7. *For every $v^* \in R'$ the neighbourhood of v^* in $\cup\{Q(u)\colon u \in M'(v)\}$ contains a clique C^+ which r-dominates all vertices of $M'(v)$.*

Proof. We will build C^+ iteratively; initially, C^+ can be an arbitrary vertex of $\cup\{Q(u)\colon u \in M'(v)\}$. Consider the current clique C^+ and suppose that $d(w, C^+) > r(w)$ for some vertex $w \in M'(v)$. Take a gate w^* of w at distance 2 from v^*, and, let $s \in Q(w)$ be a common neighbour of w^* and v^*. Suppose that s is not adjacent to some vertex $t \in C^+$, otherwise just add s to C^+. Let $t \in Q(u)$ for a vertex $u \in M'(v)$, namely, $t \in I(u^*, v^*)$, where u^* is a gate of u. Pick two vertices $x \in C(u)$ and $z \in C(w)$. Since $d(w, t) > r(w)$, we conclude that $tw^* \notin E$, in particular, $t \notin Q(w)$. We assert that $s \notin Q(u)$, for otherwise s and x will be adjacent. To avoid a forbidden house induced by s, z, x, t and v^* the vertices z and t must be adjacent (recall that $v^*x, v^*z \notin E$). But then we have created a house w^*, z, s, t, v^*. Hence $s \notin Q(u)$, and Claim 5 yields that $C(u) \cap C(w) = \emptyset$; see Fig. 5 for an illustration.

In the obtained 5-cycle (t, v^*, s, z, x, t) the vertices x, s and z, t must be adjacent. Since the vertices t, z, w^*, s, v^* cannot induce a house and $st \notin E$, the vertices w^* and t are adjacent, which is impossible. This proves that if $w \in M'(v)$ is still non-dominated by C^+, then we can extend C^+ by adding an arbitrary common neighbour $s \in Q(w)$ of w^* and v^*. □

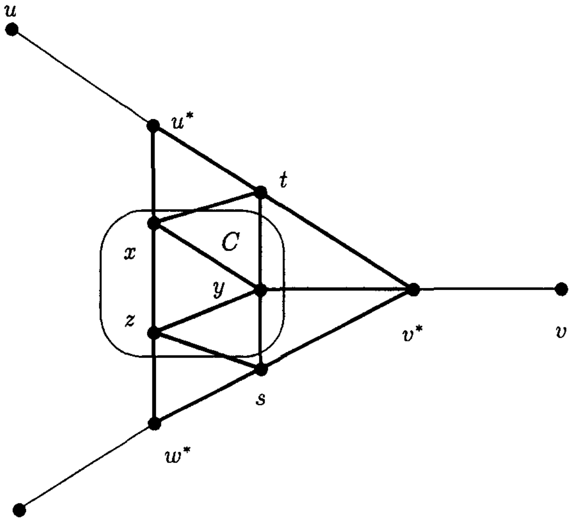


Fig. 5.

Claim 8. *If both $M'(v)$ and $M''(v)$ are non-empty, then $R' \cap R'' \neq \emptyset$, i.e., there exists a gate v^* of v which belongs to all sets $R(u), u \in M'(v)$, and is adjacent to at least one gate of each vertex of $M''(v)$.*

Proof. Assume that $R' \cap R'' = \emptyset$ and consider arbitrary vertices $v_1^* \in R'$ and $v_2^* \in R''$. Choose the vertices $u \in M'(v)$ and $w \in M''(v)$ such that $v_2^* \notin R(u)$ and v_1^* is not adjacent to any gate of w. Let u^* and w^* be gates of w and u such that $d(u^*, v_1^*) = 2$ and $d(w^*, v_2^*) = 1$; see Claims 3 and 4. Pick an arbitrary vertex $x \in C(u)$. From Claim 2 we know that $x \in C(w)$, i.e., x is a common neighbour of u^* and w^*; see Fig. 6. Let $t \in Q(u)$ be a common neighbour of u^* and v_1^*. Since G is house-free and $v_2^* \notin R(u)$, this implies that t is adjacent to w^* and is not adjacent to v_2^*.

The vertices t, v_1^*, w^*, v_2^*, y cannot induce a house. The vertices w^* and v_1^* cannot be adjacent, because v_1^* is not adjacent to any gate of w. Therefore v_1^* and v_2^* are adjacent, and we obtain a house induced by x, t, w^*, v_1^*, v_2^*. □

Now, we have all prerequisites to accomplish the proof of the theorem. We have to construct a new clique C^* which r-dominates all vertices of M^* and is at distance $k-1$ from the vertex $v \in M - M^*$. First of all, note that every vertex of $M^* - M^+(v)$ is r-dominated by the clique $C(v)$. So, if $M^+(v) = \emptyset$, just define $C^* = C(v) \cup \{v^*\}$, where v^* is an arbitrary gate of v. If $M^+(v) = M''(v)$, the clique C^* is defined in a similar way, but in this case v^* is choosen from R''; see Claim 3 and the definition of the set R''. Otherwise, if $M^+(v) = M'(v)$, then put $C^* = C^+ \cup C(v) \cup \{v^*\}$, where $v^* \in R'$ and C^+ is an r-dominated clique of $M'(v)$ described in Claim 7. Finally, if both $M'(v)$ and $M''(v)$ are non-empty, the clique C^* has the same form as in the preceding case, but v^* is selected from $R' \cap R''$; see Claim 8. That C^* r-dominates M^* follows easily,

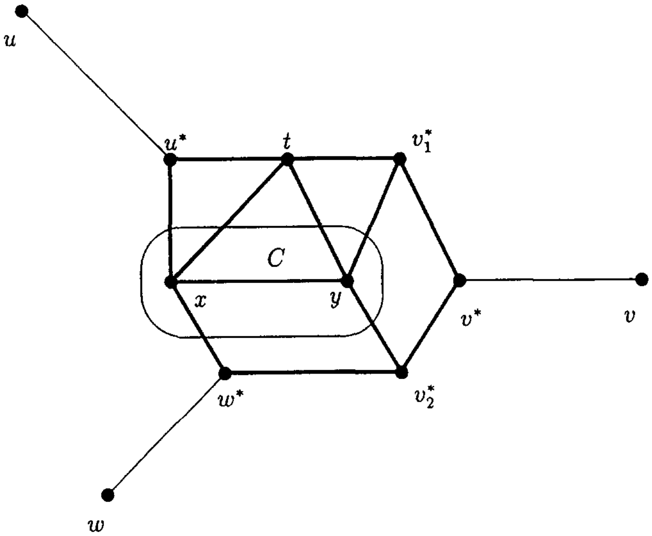


Fig. 6.

because C^+ r-dominates the vertices of $M'(v)$, the vertex v^* r-dominates the vertices of $M''(v)$, and, finally, $C(v)$ r-dominates the vertices from $M^* - M^+(v)$. Since in all cases C^* contains a gate v^* of v, we obtain that $d(v, C^*) < k = d(v, C)$. This finishes the proof of the theorem. □

From the proof we conclude that an r-dominating clique of M can be constructed in $O(n^5)$ time, where n is the number of vertices of the graph G. Indeed, as noted above, there are at most $\sum_{v \in M} d(v, C_0)$ iterations, where C_0 is the starting clique. At each iteration a new current clique C^* of the form $C(v) \cup \{v^*\}$ or $C^+ \cup C(v) \cup \{v^*\}$ is returned. A straightforward analysis of the proof shows that with a distance matrix in hand the sets $C(v)$ and C^+, as well as R', R'' and $R' \cap R''$ can be computed in polynomial time $O(n^3)$.

Remark. The problem of describing all graphs such that any finite subset M has an r-dominating clique provided that $d(u, v) \leqslant r(u) + r(v) + 1$ holds for all $u, v \in M$, seems to be close to that of characterization of absolute retracts of reflexive graphs alias Helly graphs [6,22,23].

Acknowledgements

I am indebted to one referee for helpful remarks about the presentation and to another referee for a careful reading and helpful suggestions.

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