Embedding into the Rectilinear Grid

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Abstract: We show that the embedding of metric spaces into the l_1-grid $\mathbb{Z}^2$ can be characterized in essentially the same fashion as in the case of the l_1-plane $\mathbb{R}^2$. In particular, a metric space can be embedded into $\mathbb{Z}^2$ iff every subspace with at most 6 points is embeddable. Moreover, if such an embedding exists, it can be constructed in polynomial time (for finite spaces). © 1998 John Wiley & Sons, Inc.
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The rectilinear metric (alias l_1-metric) is probably the simplest distance measure in $\mathbb{R}^n$. This explains why in many cases rectilinear spaces are selected as host spaces for embedding a given metric space; we refer to Hubert et al. [7] for an application to multidimensional scaling where one aims at producing a visual display of a given data set and to Deza and Laurent [6] for other applications. Surprisingly, the problem to characterize the metric subspaces of the rectilinear space of a given dimension seems to be much more difficult than is the analogous problem for Euclidean spaces, where suitable criteria are given by the classical results of Menger and Schoenberg (see [4]). However, at least the two-dimensional case can be approached. Recently, by sharpening a result of Malitz and Malitz [8], we proved the following Menger-type theorem for the rectilinear plane $\mathbb{R}^2$: *A metric space is embeddable in the plane if and only if every subspace with at most* 6 *points is such* [1]. It was the purpose of this paper to demonstrate that the same result holds true for the embedding in the grid $\mathbb{Z}^2$ (''digital plane'') endowed with the rectilinear distance (alias city block metric) (see Fig. 1). Again, as in [1], we can take advantage of such notions as ''d-split'' and ''totally decomposable metrics,'' introduced and investigated in the case of finite metric spaces by Bandelt and Dress [2].

Let (X, d) be a metric space. We will say that (X, d) satisfies the *parity condition* if $d(u, v) + d(v, w) + d(w, u)$ is an even integer for any $u, v, w \in X$ (cf. [10] for a straightforward metric characterization of bipartite graphs). To every pair $S = \{A, B\}$ of nonempty subsets of X we associate the *isolation index* α_S (if it exists) with respect to d, as follows:

$$\alpha_S = \tfrac{1}{2}\cdot \min_{\substack{a,a'\in A\\ b,b'\in B}} (\max\{d(a, b) + d(a', b'), d(a, b') + d(a', b), d(a, a') + d(b, b')\} - d(a, a') - d(b, b')).$$

Evidently, if (X, d) satisfies the parity condition, then all distances are integers, and, hence, for every pair $S = \{A, B\}$, the minimum is attained, so that $\alpha_S \geq 0$ is well

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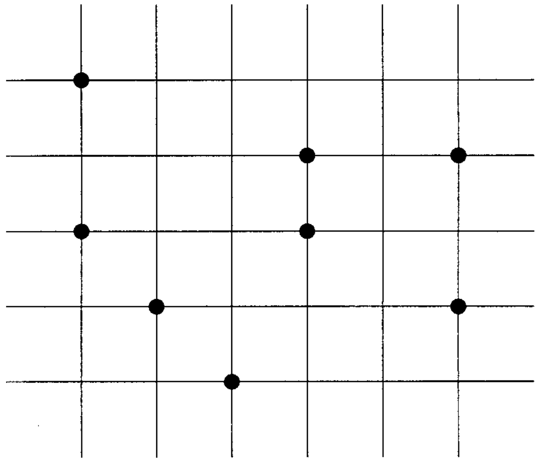

Fig. 1. A finite subspace of the l_1-grid $\mathbb{Z}^2$.

defined. An unordered pair $S = \{A, B\}$ for which α_S exists and is positive is called a *partial d-split.* In this case, A and B are necessarily disjoint. The pair $S = \{A, B\}$ is a *d-split* if, in addition, $A \cup B = X$. Either part A, B is then called a *d-half-space* of X. Following [2], a finite metric space (X, d) is called *totally decomposable* if

$$d = \sum \alpha_S \cdot \delta_S,$$

where the sum extends over all d-splits and δ_S denotes the *split metric* associated with the d-split $S = \{A, B\}$:

$$\delta_S(x, y) = \begin{cases} 0 & \text{if } x, y \in A \text{ or } x, y \in B, \\ 1 & \text{otherwise, i.e., if } S \text{ separates } x \text{ and } y. \end{cases}$$

For a metric space (X, d) satisfying the parity condition, the *split hypergraph* $\mathcal{SPL}(X, d)$ has all d-splits as its vertices and its edge set comprises all ''incompatible'' pairs and ''asteroidal'' triplets of d-splits, defined as follows: Two d-splits $S_1 = \{A_1, B_1\}$ and $S_2 = \{A_2, B_2\}$ are *incompatible* if all four intersections $A_i \cap B_j$ $(i, j = 1, 2)$ are nonempty. Three distinct d-splits $\{A_i, B_i\}$ $(i = 1, 2, 3)$ are said to form an *asteroidal triplet* if they have some pairwise disjoint half-spaces, say, $A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \emptyset$. This intersection pattern of d-splits is characteristic for the 3-star $K_{1,3}$ (regarded as a metric space). We say that the hypergraph $\mathcal{SPL}(X, d)$ is k-colorable if there exists a mapping from the vertex set into the set $\{1, 2, \ldots, k\}$ of ''colors'' such that no edge is monochromatic, that is, not all vertices of any edge receive the same color. The *chromatic number* is the smallest number k for which there is a k-coloring (for further information concerning hypergraphs and their colorings, consult the book of Berge [3]).

Theorem. *For a metric space (X, d) the following statements are equivalent:*

(1) (X, d) can be embedded into the l_1-grid $\mathbb{Z}^2$.
(2) Every subspace of (X, d) with at most six points can be embedded into the l_1-grid $\mathbb{Z}^2$.
(3) (X, d) satisfies the parity condition, and every subspace of (X, d) with five or six points can be embedded into the l_1-plane $\mathbb{R}^2$.
(4) Every finite subspace (Y, d) of (X, d) satisfies the parity condition and is totally decomposable such that the split hypergraph $\mathcal{SPL}(Y, d)$ is bicolorable.
(5) (X, d) satisfies:

(a) the parity condition;
(b) $d(x, y) = \sum \alpha_S \cdot \delta_S(x, y)$ for all $x, y \in X$, where each sum extends over all d-splits S and has only finitely many nonzero summands;
(c) the split hypergraph $\mathcal{SPL}(X, d)$ is bicolorable.

To prove the theorem, we commence by establishing some auxilliary results:

Lemma 1. *A metric space (X, d) can be embedded into $\mathbb{Z}$ if and only if (X, d) satisfies conditions (5a, b) of the theorem and $\mathcal{SPL}(X, d)$ is monochromatic.*

Proof. Every subspace (X, d) of $\mathbb{Z}$ can be viewed as a path the edges of which are defined by consecutive points of X. It is easy to see that any d-split of (X, d) corresponds to the pair of connected components of this path obtained by removing a suitable edge. The isolation index of the d-split equals the length of the removed edge. Therefore, $\mathcal{SPL}(X, d)$ is edgeless, that is, monochromatic. In addition, (X, d) satisfies condition (5b) of the theorem since $d(x, y)$ can be written as the sum of the isolation indices of d-splits separating x and y. The parity condition is trivially satisfied.

Conversely, pick any d-split $\{A_0, B_0\}$ of a space (X, d) having at least three points and satisfying the requirements of the lemma. There exists a minimal d-half-space A_1 properly including A_0, say, for, otherwise, the finiteness condition of (5b) would be violated. If there was yet another minimal d-half-space A_1' with this property, then $\{A_1, B_1\}$ and $\{A_1', B_1'\}$ either would be incompatible or $\{A_0, B_0\}$, $\{A_1, B_1\}$, $\{A_1', B_1'\}$ would form an asteroidal triplet. Continuing iteratively in both directions, we can establish a (possibly two-sided infinite) linear order $\{\{A_i, B_i\} \mid i \in I\}$ of d-splits (yielding a copy of some interval I of $\mathbb{Z}$). This comprises all d-splits in view of the finiteness assumption in (5b). For any $i \in I$ with $i + 1 \in I$, the d-half-spaces A_{i+1} and B_i share a single point x_i. In addition, if I has a minimal element p or a maximal element q or both of them, then the d-half-spaces A_p and B_q consist of one point x_{p-1} and x_q each. The consecutive points x_{i-1} and x_i of X are separated only by one d-split $\{A_i, B_i\}$. Applying condition (5b), we infer that the isolation index of this d-split equals $d(x_{i-1}, x_i)$. More generally, since the points x_i and x_j $(i < j)$ are separated by exactly $j - i$ d-splits $\{A_{i+1}, B_{i+1}\}, \ldots, \{A_j, B_j\}$, we deduce that $d(x_i, x_j) = d(x_i, x_{i+1}) + \cdots + d(x_{j-1}, x_j)$. Therefore, the total order on X induced by I is compatible with the metric betweeness, that is, $i \le k \le j$ if and only if $d(x_i, x_j) = d(x_i, x_k) + d(x_k, x_j)$. This yields the following isometric embedding ζ of (X, d) into $\mathbb{Z}$. Put $\zeta(x_0) = 0$, and for any other point $x_i \in X$, let

$$\zeta(x_i) = \begin{cases} d(x_0, x_i) & \text{if } i > 0, \\ -d(x_0, x_i) & \text{if } i < 0. \end{cases}$$

Then, ζ isometrically maps X into $\mathbb{R}$. The parity condition implies that d takes only integer values. Hence, ζ maps X into $\mathbb{Z}$, concluding the proof. ∎

Lemma 2. *Let (X, d) be a metric space satisfying the parity condition such that every 5-point subspace is totally decomposable. Then, every partial d-split extends to a d-split with an integer-valued isolation index, and for each pair of points x, y, there are at most $d(x, y)$ distinct d-splits separating x and y.*

Proof. First observe that all distances are integers because $d(y, z) + d(z, y) + d(y, y) = 2d(y, z)$ is an even integer for all y, z by the parity condition. Since d-splits are defined by a four-point condition, every partial d-split extends to some maximal partial d-split $\{A, B\}$ by virtue of Zorn's lemma. For $u, v \in A$ and $w, x \in B$,

$$\begin{aligned} &d(u, x) + d(v, w) - d(u, v) - d(w, x) \\ &\quad = (d(u, v) + d(u, x) + d(v, x)) \\ &\qquad + (d(v, w) + d(v, x) + d(w, x)) \\ &\qquad\qquad - 2(d(u, v) + d(v, x) + d(w, x)) \end{aligned}$$

is an even integer. Hence, all numbers over which the minimum is taken for α_S are integers; hence, the minimum is attained. If there exists $z \in X - (A \cup B)$, then neither $\{A \cup \{z\}, B\}$ nor $\{A, B \cup \{z\}\}$ would be a partial d-split, but then there are $a_1, a_2, a_3 \in A$ and $b_1, b_2, b_3 \in B$ such that the partial d-split $\{\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}\}$ would not extend to a d-split on $\{a_1, a_2, a_3, b_1, b_2, b_3\ z\}$, contrary to the fact that all finite subspaces are totally decomposable. (Recall that totally decomposability is fully described by a 5-point condition [2].) On every finite subset of X containing x, y there are no more than $d(x, y)$ distinct d-splits separating x and y. Hence, the same holds for the whole set X. ∎

Proof of the Theorem. (1) ⇒ (2) and (2) ⇒ (3) are trivial.

(3) ⇒ (4). According to the Menger-type theorem for the rectilinear plane established in [1], the space (X, d) can be embedded into the l_1-plane $\mathbb{R}^2$. Therefore, any finite subspace (Y, d) of (X, d) being a subspace of $\mathbb{R}^2$ must be totally decomposable. To prove that $\mathcal{SPL}(Y, d)$ is bicolorable, consider the smallest isothetic rectangle containing the set Y. Transform this rectangle into a rectilinear grid G by taking all intersection points of vertical and horizontal lines passing through the points of Y. Then, G endowed with the rectilinear distance is a totally decomposable space. The split hypergraph $\mathcal{SPL}(G, \|\cdot\|_1)$ of this space is bicolorable. Indeed, any d-split of G can be obtained by removing the edges contained in one and the same horizontal or vertical strip of G, and, accordingly, we refer to the d-split as being horizontal or vertical, respectively. The d-splits of G obtained by removing parallel edges of G evidently induce an edgeless subhypergraph of $\mathcal{SPL}(G, \|\cdot\|_1)$; therefore, the latter is bicolorable. Since (Y, d) is a subspace of a totally decomposable space $(G, \|\cdot\|_1)$, any d-split $S = \{A, B\}$ of Y extends to a d-split $S' = \{A', B'\}$ of G [2]; in fact, S may have two extensions, to a vertical d-split as well as to a horizontal d-split. Then, assigning to S the color of S' and breaking ties arbitrarily, we obtain a correct bicoloring of the hypergraph $\mathcal{SPL}(Y, d)$. Indeed, every pair of incompatible d-splits of Y extends to a pair of incompatible d-splits of G, while every asteroidal triplet of Y extends to three d-splits of G among which at least two are incompatible.

(4) ⇒ (5). Lemma 2 guarantees that each pair of points x, y can be separated by only a finite number of d-splits. For every such d-split $S = \{A, B\}$ with $x \in A$ and $y \in B$, select one quartet of points $a, a' \in A$ and $b, b' \in B$ for which the minimum for α_S is attained. Let Y denote the union of x, y and the associated quartets a, a', b, b' over all d-splits of X separating the points x and y. As (Y, d) is totally decomposable, we can write

$$d(x, y) = \sum \alpha_{S'} \delta_{S'}(x, y),$$

where the sum extends over all d-splits $S' = (A', B')$ of Y separating x and y. By Lemma 2, each d-split $S' = (A', B')$ of Y extends to a d-split $S = (A, B)$ of X; hence, $\alpha_S \le \alpha_{S'}$. On the other hand, as $a, a' \in A \cap Y$ and $b, b' \in B \cap Y$, we obtain the converse inequality $\alpha_{S'} \le \alpha_S$. Therefore,

$$d(x, y) = \sum \alpha_S \cdot \delta_S(x, y),$$

where the sum extends over all d-splits of (X, d), thus establishing (5b).

It remains to show that the split hypergraph $\mathcal{SPL}(X, d)$ is bicolorable. Recall that the compactness theorem of propositional logic guarantees that a hypergraph is k-colorable if and only if every finite subhypergraph is k-colorable. (Here, a *subhypergraph* is understood to be induced by a subset M of vertices of a hypergraph $\mathcal{H}$, i.e., obtained by taking all intersections of M with edges of $\mathcal{H}$.) Now, if $\mathcal{SPL}(X, d)$ is not bicolorable, we can find a finite subhypergraph $\mathcal{F}$ of $\mathcal{SPL}(X, d)$ which is not bicolorable either. We assert that $\mathcal{F} \subseteq \mathcal{SPL}(Y, d)$ for some finite set $Y \subseteq X$. To construct Y, we proceed as follows: First, if $S_1 = \{A_1, B_1\}$ and $S_2 = \{A_2, B_2\}$ are two d-splits of $\mathcal{F}$, then we include in Y one point from each nonempty intersection $A_1 \cap A_2$, $A_1 \cap B_2$, $B_1 \cap A_2$, and $B_1 \cap B_2$. Since the restriction to Y of any d-split of (X, d) is a d-split of (Y, d), we infer that any edge of $\mathcal{F}$ (incompatible pair or asteroidal triplet) restricts to an edge of the hypergraph $\mathcal{SPL}(Y, d)$, that is, $\mathcal{F}$ is isomorphic to $\mathcal{SPL}(Y, d)$. This implies that $\mathcal{SPL}(Y, d)$ cannot be bicolored, contrary to our assumption.

(5) ⇒ (1). Let $\mathcal{F}'$ and $\mathcal{F}''$ be the color classes of the hypergraph $\mathcal{SPL}(X, d)$, and let

$$d' = \sum_{S \in \mathcal{F}'} \alpha_S \cdot \delta_S, \quad d'' = \sum_{S \in \mathcal{F}''} \alpha_S \cdot \delta_S.$$

Then, d' and d'' are well defined such that $d(x, y) = d'(x, y) + d''(x, y)$ for all $x, y \in X$, because of condition (5b). Next, we show that the metric spaces (X, d') and (X, d'') fulfill the parity condition. From the proof of Lemma 2, we know that d-splits of a metric space satisfying the parity condition have only integer-valued isolation indices. Hence, both d' and d'' take only integral values. For any points $a, b, c \in X$, if $d'(a, c) \geq \max\{d'(a, b), d'(b, c)\}$, necessarily $d'(a, c) = d'(a, b) + d'(b, c)$, because, otherwise, $\mathcal{F}'$ would not be edgeless. This implies that $d'(a, b) + d'(b, c) + d'(c, a)$ is an even integer, and, therefore, (X, d') satisfies the parity condition. By Lemma 1, either of the metric spaces (X, d') and (X, d'') is embeddable into $\mathbb{Z}$. As $d = d' + d''$, we deduce that (X, d) can be embedded into the l_1-grid $\mathbb{Z}^2$. This concludes the proof of the theorem. ■

Corollary 1. *For a subspace (X, d) of the l_1-plane $\mathbb{R}^2$, the following conditions are equivalent:*

(1) *(X, d) can be embedded into the l_1-grid $\mathbb{Z}^2$;.*
(2) *The isolation indices of d-splits are integers;.*
(3) *(X, d) satisfies the parity condition..*

From [8] we know that an n-point metric space (X, d) can be tested in $O(n^3)$ time whether or not it embeds into the l_1-plane $\mathbb{R}^2$ and we can construct such an embedding if it exists. From this and condition (3) of Corollary 1, we immediately conclude that the metric spaces embedable into the l_1-grid $\mathbb{Z}^2$ can be recognized within the same time bounds. However, one can find an isometric embedding into the rectilinear grid much faster as soon as such an embedding into the rectilinear plane is given. Next, we present an $O(n \log n)$ time algorithm which transforms a given embedding of (X, d) in $\mathbb{R}^2$ into an embedding in $\mathbb{Z}^2$ if the latter exists.

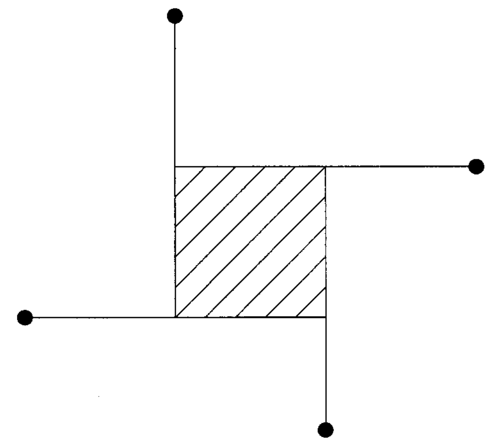

Fig. 2. The efficient set of four points (in generic position).

Consider the grid G within the smallest isothetic rectangle containing X (regarded as a subset of $\mathbb{R}^2$) as in the proof of implication (3) ⇒ (4) of the theorem. As was mentioned there, every d-split of (X, d) extends to at most two d-splits of G. In the ideal case when every d-split $S = \{A, B\}$ of (X, d) has a unique extension $S' = \{A', B'\}$, then $\alpha_S = \alpha_{S'}$ equals the width of the corresponding strip of G. (To calculate all widths, we have to sort the coordinates of the points of X.) Now, if there is a strip with nonintegral width, then (X, d) cannot be embeddable into the rectilinear grid in view of condition (3) of Corollary 1. Otherwise, if all widths are integers, then (X, d) is embeddable in the l_1-grid $\mathbb{Z}^2$. To obtain such an embedding, pick a point $x \in X$ and translate the whole set X in the direction xO, where $O = (0, 0)$.

Now consider the case when a d-split $S = \{A, B\}$ of (X, d) can be extended to a vertical d-split $S' = \{A', B'\}$ and a horizontal d-split $S'' = \{A'', B''\}$ of G. Since $A = (A' \cap A'') \cap X$ and $B = (B' \cap B'') \cap X$, necessarily A and B belong to disjoint quadrants incident with opposite corners of some rectangular cell R of the grid G. Then, $\alpha_S = \alpha_{S'} + \alpha_{S''}$, that is, α_S is the semiperimeter of the rectangle R. We will show how to find all such rectangles R and corresponding d-splits $S = (A, B)$ of (X, d) without explicit knowledge of the grid G [which in the worst case would contain $O(n^2)$ vertices].

A point $p \in \mathbb{R}^2$ is said to be an *efficient* point of X if there does not exist any other point $q \in \mathbb{R}^2$ such that $\|q - x\|_1 \leq \|p - x\|_1$ for $x \in X$ and $\|q - y\|_1 < \|p - y\|_1$ for at least one $y \in X$. Denote the set of all efficient points by $Eff(X)$, called the *efficient set* of X; for an illustration, see Figure 2. An $O(n \log n)$ time algorithm to compute

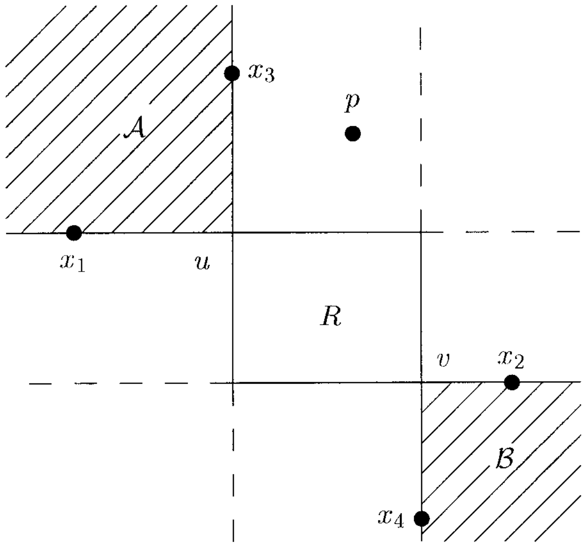

Fig. 3. Decomposition of X.

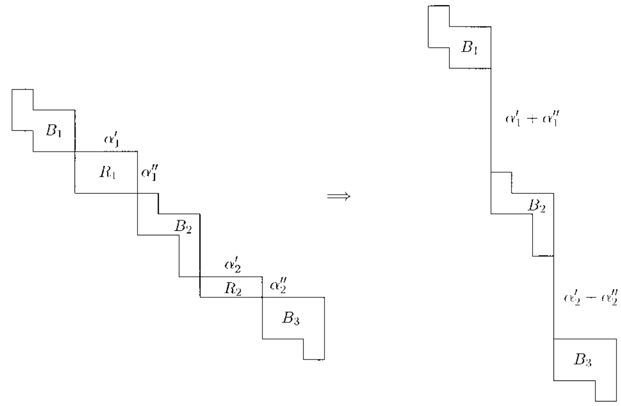

Fig. 4. Transformation of an embedding in the l_1-plane.

the efficient set of n points in the l_1-plane is presented in [5] (for properties of $Eff(X)$ and an $O(n^2)$ algorithm, see also [11]). $Eff(X)$, being ortho-convex, is a union of ortho-convex (possibly degenerated) rectilinear polygons glued together along vertices [they become cut points of $Eff(X)$]. Having the description of $Eff(X)$ as an output of the algorithm from [5], one can easily generate all blocks and cut points of $Eff(X)$. Of special interest for us are the rectangular blocks R that are incident with exactly two other blocks via opposite corners u and v of R such that $R \cap X \subseteq \{u, v\}$. In this case, the whole set X is contained in two disjoint quadrants defined by u and v. Moreover, R is a cell of the grid G. Indeed, according to the algorithm of [5], every boundary edge of $Eff(X)$ contains at least one point from X. Conversely, we assert that any rectangular cell R of G with the property that the horizontal and vertical d-splits of G crossing its sides extend one and the same d-split $S = \{A, B\}$ of (X, d) corresponds to such a block of $Eff(X)$. It suffices to establish that $R \subseteq Eff(X)$ and that $Eff(X) - R$ belongs to the same quadrants $\mathcal{A}$ and $\mathcal{B}$ as the d-half-spaces A and B. For this purpose, recall the characterization of efficient points given in [5]: *$p \in \mathbb{R}^2$ is an efficient point of X if and only if for every $x \in X$ there is a point $y \in X$ such that p belongs to a shortest l_1-path connecting x and y*. Since this condition is fulfilled by any point $p \in R$ and $x \in A$, $y \in B$, we obtain that R consists of efficient points only. Now, let $p \notin \mathcal{A} \cup \mathcal{B}$. Since R is a cell of G, each of the horizontal and vertical rays bounding the sets $\mathcal{A}$ and $\mathcal{B}$ has at least one point in common with X. Select such points and denote them by x_1, x_2, x_3, and x_4 as indicated in Figure 3 (x_1, x_3 and x_2, x_4 may coincide with u and v, respectively). It is easy to see that independent of the position of p outside $\mathcal{A} \cup \mathcal{B}$ the above condition is violated by some point x selected from x_1, x_2, x_3, x_4 (for the situation depicted in Fig. 3 this would be the point x_1). We conclude that $Eff(X)$ consists of such connecting rectangles $R_1, \ldots, R_m$, and the other blocks $B_1, \ldots, B_k$ which cover X. We construct a new isometric embedding of (X, d) into the l_1-plane $\mathbb{R}^2$ as is sketched in Figure 4: We maintain the embedding for points of X inside every block B_j, $j = 1, \ldots, k$, and successively substitute every rectangle R_i by a vertical segment of length equal to the semiperimeter of R_i, which joins translates of the blocks B_i and B_{i+1}. Evidently, all this can be done in $O(n \log n)$ time.

The preceding results for the rectilinear digital metric immediately apply to the other classical digital metric (cf. [9]), viz., the ''chessboard'' (or ''king's'') metric induced by the $\|\cdot\|_\infty$ norm. Indeed, it is well known that there is an isometry from the l_1-plane to the l_∞-plane: it suffices to rotate the plane by 45° and then shrink it by a factor $\frac{1}{\sqrt{2}}$. A similar procedure transforms the l_1-grid into the l_∞-grid (''king's grid'') (Fig. 5). By turning the king's grid by 45° and inserting new crossing points (creating edges of length $\frac{1}{2}$ and removing the superfluous edges of length 1), we obtain the l_1-grid $(\frac{1}{2}\mathbb{Z})^2$.

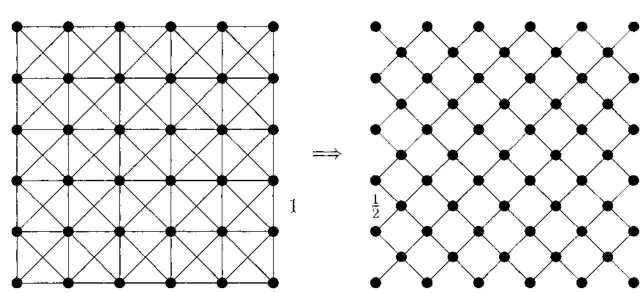

Fig. 5. From the l_∞-grid to the l_1-grid.

Corollary 2. *A metric space* (X, d) *is embeddable in the* l_∞*-grid* $\mathbb{Z}^2$ *if and only if* d *is integer-valued and* $(X, 2d)$ *is embeddable in the* l_1*-grid* $\mathbb{Z}^2$.

REFERENCES

[1] H.-J. Bandelt and V. Chepoi, Embedding metric spaces in the rectilinear plane: A six-point criterion. *Discr. Comput. Geom.* **15** (1996) 107–117.

[2] H.-J. Bandelt and A. W. M. Dress, A canonical decomposition theory for metrics on a finite set. *Adv. Math.* **92** (1992) 47–105.

[3] C. Berge, *Hypergraphs.* North-Holland (1989).

[4] L. M. Blumenthal, *Theory and Applications of Distance Geometry.* Oxford University Press (1953).

[5] L. G. Chalmet, R. L. Francis, and A. Kolen, Finding efficient solutions for rectilinear distance location problems efficiently. *Eur. J. Oper. Res.* **6** (1981) 117–124.

[6] M. Deza and M. Laurent, *Geometry of Cuts and Metrics.* Springer-Verlag, Berlin (1997).

[7] L. J. Hubert, P. Arabie, and M. Hesson-Mcinnis, Multidimensional scaling in the city-block metric: A combinatorial approach. *J. Classif.* **9** (1992) 211–236.

[8] S. M. Malitz and J. I. Malitz, A bounded compactness theorem for L^1-embeddability of metric spaces in the plane. *Discr. Comput. Geom.* **8** (1992) 373–385.

[9] R. A. Melter, A survey of digital metrics. *Contemp. Math.* **119** (1991) 95–106.

[10] R. Melter and I. Tomescu, Isometric embeddability for graphs. *Ars Combin.* **12** (1981) 111–115.

[11] R. E. Wendell, A. P. Hurter, and T. J. Lowe, Efficient points in location problems. *AIIE Trans.* **9** (1977) 238–246.