4. K. V. Rudakov, "Completeness and universal constraints in the problem of correction of heuristic classification algorithms," Kibernetika, No. 3, 106-109 (1987).
5. K. V. Rudakov, "Symmetry and function constraints in the problem of correction of heuristic classification algorithms," Kibernetika, No. 4, 74-77 (1987).
6. K. V. Rudakov, On Some Classes of Recognition Algorithms (General Results) [in Russian], VTs AN SSSR, Moscow (1980).

ISOMETRIC SUBGRAPHS OF HAMMING GRAPHS AND d-CONVEXITY

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In this study, we provide criteria of isometric embeddability of graphs in Hamming graphs. We consider ordinary connected graphs with a finite vertex set endowed with the natural metric d(x, y), equal to the number of edges in the shortest chain between the vertices x and y.

Let $a_1, \ldots, a_n$ be natural numbers. The Hamming graph $H_{a_1 \ldots a_n}$ is the graph with the vertex set $X = \{x = (x_1, \ldots, x_n) : 1 \leqslant x_i \leqslant a_i,\ i = \overline{1, \ldots n}\}$ in which two vertices are joined by an edge if and only if the corresponding vectors differ precisely in one coordinate [1, 2]. In other words, the graph $H_{a_1 \ldots a_n}$ is the Cartesian product of the graphs $H_{a_1}, \ldots, H_{a_n}$, where H_{a_i} is the a_i-vertex complete graph. It is easy to show that in the Hamming graph the distance d(x, y) between the vertices x, y is equal to the number of different pairs of coordinates in the tuples corresponding to these vertices, i.e., it is equal to the Hamming distance between these tuples. It is also easy to show that the graph of the n-dimensional cube Q_n may be treated as the Hamming graph $H_{2 \ldots 2}$.

We say that the set $A \subset Y$ in the metric space (Y, d) is isometrically embeddable in the metric space (Y_0, d_0) if there exists a mapping α from A to Y_0 such that for any $x, y \in A$ we have $d(x, y) = d_0(\alpha(x), \alpha(y))$. The graph G = (X, U) is called an isometric subgraph G_0 if the vertex set of X of G is isometrically embeddable in G_0. If the graph G is an isometric subgraph of some Hamming graph, we say that G is isometrically embeddable in the Hamming graph.

The set $M \subset Y$ of the metric space (Y, d) is called d-convex if for any $x, y \in M$ the line segment $\langle x, y\rangle = \{z : d(x, y) = d(x, z) + d(z, y)\}$ is contained in M (see, e.g., [3]). A half-space is a d-convex set with a d-convex complement. Also recall [4] that the set $M \subset Y$ is called r-convex if for any $x, y \in M$ such that $\langle x, y\rangle \neq \{x, y\}$ there exists a point $z \in \langle x, y\rangle \cap M$ distinct from x, y.

For the set $M \subset Y$ and the point x in the space (Y, d), we denote by $N_x(M) = \{z \in M : d(x, z) = d(x, M) = \inf\{d(x, u) : u \in M\}\}$ the metric projection of x on M. If for any $x \in Y$ the set $N_x(M)$ is a one-point set, then M is a Chebyshev set (see, e.g., [5]). For the point $x \in M$, let $W_x(M) = \{z \in Y : N_z(M) = \{x\}\}$. Also let $W(M) = \{z \in Y : N_z(M) = M\}$.

The interest in isometric embedding of graphs in Hamming graphs was stimulated by the work of Graham and Pollak [6, 7]. Isometric subgraphs of hypercubes were described by Djokovic [8] (this problem is also considered in [9] in the context of various location problems). Concerning some applications of Djokovic's criterion and other topics linked with isometric embedding of special metric spaces in hypercubes, see [10, 11]. The question of explicit description of isometric subgraphs of Hamming graphs was considered and partially solved by Winkler [1].

The description of isometric subgraphs of Hamming graphs or hypercubes also can be approached differently. Consider the integral lattice $Z^n = Z \times \ldots \times Z$ equipped with one of the metrics

$$d_1(x, y) = \sum_{i=1}^{n} \operatorname{sign} |x_i - y_i|, \quad d_2(x, y) = \sum_{i=1}^{n} |x_i - y_i|,$$

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where $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$. The corresponding metric spaces will be denoted by (Z^n, d_1), (Z^n, d_2). Then the problem of characterization of isometric subgraphs of Hamming graphs is equivalent to the problem of describing finite r-convex sets of the spaces (Z^n, d_1), $n \geqslant 1$, and all r-convex sets from (Z^n, d_2) are isometric subgraphs of hypercubes. Both these questions are a specialization of the following problem: describe the r-convex sets (i.e., isometric subgraphs) of the spaces of the form (Z^n, d), where d is an integral metric on the lattice Z^n. In this respect, note that Yushmanov's results [12] imply that any finite graph is isometrically embeddable in some space of the form (Z^n, d_3), where $d_3((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max(|x_1 - y_1|, \dots, |x_n - y_n|)$. The notions of coordinate and diagonal convexity are used in [13] to describe isometric subgraphs of the lattices (Z^2, d_2) and (Z^2, d_3).

LEMMA 1. For any edge $e = (x, y)$ of the Hamming graph H, the sets $W_x(e)$, $W_y(e)$, $W(e)$ are half-spaces.

Proof. Let $H = H_{a_1\dots a_n}$ and let the tuples $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n)$ differ in the i-th coordinate. Let $H_0 = \prod_{j\neq i}\{1, \dots, a_j\}$. Then it is easy to see that

$$W_x(e) = H_0 \times \{x_i\},\quad W_y(e) = H_0 \times \{y_i\},$$
$$W(e) = H_0 \times \{\{1, \dots, a_i\}\setminus\{x_i, y_i\}\},$$
$$W_x(e) \cup W_y(e) = H_0 \times \{x_i, y_i\},\quad W_x(e) \cup W(e) =$$
$$= H_0 \times \{\{1, \dots, a_i\}\setminus\{y_i\}\},\quad W_y(e) \cup W(e) = H_0\ \{\{1, \dots, a_i\}\setminus\{x_i\}\}.$$

We know [14] that the set $A = A_1 \times \dots \times A_m$ is d-convex in the space $X = X_1 \times \dots \times X_m$ equipped with the metric $d((x_1, \dots, x_m), (y_1, \dots, y_m)) = \sum_{i=1}^{m} d_i(x_i, y_i)$ only if the sets A_i are d-convex in the spaces (X_i, d_i). Hence follows d-convexity of the sets $W_x(e)$, $W_y(e)$, $W(e)$ and their complements.

LEMMA 2. If the metric spaces $(X_i^{(1)}, d_i^{(1)})$ are isometrically embeddable in the metric spaces $(X_i^{(2)}, d_i^{(2)})$, $i = \overline{1, \dots, n}$, then the space $\left(\prod_{i=1}^{n} X_i^{(1)}, d^{(1)}\right)$ is isometrically embeddable in $\left(\prod_{i=1}^{n} X_i^{(2)}, d^{(2)}\right)$, where

$$d^{(j)}(x^j, y^j) = \sum_{i=1}^{n} d_i^{(j)}(x_i^{(j)}, y_i^{(j)}),\quad x^j = (x_1^{(j)}, \dots, x_n^{(j)}),$$
$$y^j = (y_1^{(j)}, \dots, y_n^{(j)}),\quad j = 1, 2.$$

A block graph [15] is a connected graph in which every block induces a complete subgraph.

THEOREM 1. For the connected graph $G = (X, U)$, the following conditions are equivalent:

1) G is isometrically embeddable in a Hamming graph;

2) G is isometrically embeddable in the direct product of block graphs;

3) any set $A \subset X$, $|A| \leqslant 5$, is isometrically embeddable in a Hamming graph;

4) for every edge $e = (x, y)$, the sets $W_x(e)$, $W_y(e)$, $W(e)$ are half-spaces;

5) any complete subgraph C of G satisfies the following conditions:

a) if for the vertex $z \in X$ the projection $N_z(C)$ is not a one-point set, then $N_z(C) = C$,

b) the sets $W_x(C)$, $W(C)$ are half-spaces for any $x \in C$.

Proof. The implications 1) ⇒ 3), 2) ⇒ 1) are obvious, the implication 3) ⇒ 4) follows from Lemma 1.

4 ⇒ 5). Consider a complete subgraph C of the graph G and assume that for some vertex $z \in X$ we have $x, y \in N_z(C)$. For the edge $e = (x, y)$, the set $W(e)$ is d-convex, therefore if $v \in C$, then $d(v, z) = d(z, C)$, i.e., $N_z(C) = C$.

Given condition a), d-convexity of the sets $W_x(C)$, $X\setminus W_x(C)$, $W(C)$ follows from the relations $W(C) = \cap\{W(e)\colon e = (x, y), x, y \in C\}$, $W_x(C) = W_x(e)$, $X\setminus W_x(C) = X\setminus W_x(e)$, where $e = (x, y)$ is an arbitrary edge from C with one end point in x. We will now show that the set $X\setminus W(C) = \cup\{W_v(C)\colon v \in C\}$ is d-convex. Let $z_1 \in W_x(C)$, $z_2 \in W_y(C)$ for some $x, y \in C$. For the edge $e = (x, y)$, the set $W_x(e) \cup W_y(e) = X\setminus W(e)$ is d-convex. Therefore, $\langle z_1, z_2\rangle \subset W_x(e) \cup W_y(e) \subset X\setminus W(C)$.

5 $\Rightarrow$ 1). Let Σ be the collection of all complete subgraphs from G containing at least one edge. On Σ we define the relation θ: for $C_1, C_2 \in \Sigma$, $C_1 \theta C_2$ if for every $x' \in C_1$ there exists a vertex $x'' \in C_2$ such that $W_{x'}(C_1) = W_{x''}(C_2)$. In this case, we say that the subgraphs C_1 and C_2 are comparable.

Among all the subgraphs from Σ comparable with the edge e = (x, y), select the subgraph C_0 with the maximum number of vertices. We will show that C_0 is a Chebyshev set. Let $C_0 = (x_1, \ldots, x_m)$, and $W_x(e) = W_{x_1}(C_0)$, $W_y(e) = W_{x_2}(C_0)$. If C_0 is not a Chebyshev set, then $W(C_0) \neq \emptyset$. Let v be the vertex of the set $W(C_0)$ closest to C_0. Then for any two vertices x_i, x_j from C_0 we have $\langle x_i, v\rangle \cap \langle x_j, v\rangle = \{v\}$. Let v_i be the vertex in the segment $\langle x_i, v\rangle$ adjacent to v. All the vertices $v_1, \ldots, v_m$ are distinct, and $v_i \in W_{x_i}(C_0)$, $i = \overline{1, \ldots, m}$. Since the set $X \setminus W(C_0) = \bigcup_{i=1}^{m} W_{x_i}(C_0)$ is d-convex, the set $C = \{v, v_1, \ldots, v_m\}$ induces a (m + 1)-vertex complete subgraph. In order to obtain a contradiction with the choice of the subgraph C_0, we will show that $C_0 \theta C$, i.e., $W_{x_i}(C_0) = W_{v_i}(C)$, $i = \overline{1, \ldots, m}$. First note that $x_i \in W_{v_i}(C)$. Indeed, if $x_i \in X \setminus W_{v_i}(C)$, then $x_i \in W_{v_j}(C)$ for some vertex $v_j \in C$ [the case $x_i \in W(C) \cup W_v(C)$ is impossible, since $v_i \in \langle x_i, v\rangle$]. The relations $x_j \in \langle x_i, v_j\rangle$, $v_j \in \langle x_i, v_i\rangle$ imply that $x_j \in \langle x_i, v_i\rangle$, contradicting the inclusion $v_i \in W_{x_i}(C_0)$. Now let $z \in W_{x_i}(C_0)$. If $N_z(C)$ is a one-point set, then by d-convexity of the set $W_{x_i}(C_0)$ we obtain that $N_z(C) = \{v_i\}$, i.e., $z \in W_{v_i}(C)$. Now consider the case when $N_z(C) = C$. Then $z, x_j \in X \setminus W_{v_i}(C)$, $x_i \in W_{v_i}(C) \cap \langle x_j, z\rangle$, contradicting d-convexity of the set $X \setminus W_{v_i}(C)$. We thus have the relation $W_{x_i}(C_0) \subseteq W_{v_i}(C)$. The converse inclusions are proved similarly. Thus, $C_0 \theta C$, i.e., the complete subgraph C is comparable with the edge e. This contradicts the choice of C_0, and therefore C_0 is a Chebyshev set.

Let Σ_0 be the collection of all Chebyshev sets from Σ. From the preceding proof it follows that $\Sigma_0 \neq \emptyset$. It is easy to see that on Σ_0 the relation θ is an equivalence. Therefore, the family Σ_0 is partitioned into equivalence classes $\bar{c}_1, \ldots, \bar{c}_n$. Let a_i be the number of vertices of an arbitrary subgraph from the class $\bar{c}_i$. For $\bar{c}_i$ also let $\bar{c}_i \equiv \{W_1^i, \ldots, W_{a_i}^i\}$, where $W_j^i = W_{x_j}(C)$, $j = \overline{1, \ldots, a_i}$, for any subgraph $C = \{x_1, \ldots, x_{a_i}\}$ from $\bar{c}_i$. The family $\{W_1^i, \ldots, W_{a_i}^i\}$ partitions the vertex set of the graph G into d-convex sets.

It follows from the above proof that any edge e = (x, y) is comparable with all the subgraphs of some equivalence class $\bar{c}_i$. Let $C_1 \in \bar{c}_i$. Also assume that $e \theta C_2$ for some subgraph $C_2 \in \bar{c}_j$. Let $C_1 = \{x_1, \ldots, x_{a_i}\}$, $C_2 = \{y_1, \ldots, y_{a_j}\}$, where $W_{x_1}(C_1) = W_x(e) = W_{y_1}(C_2)$, $W_{x_2}(C_1) = W_y(e) = W_{y_2}(C_2)$. Since the subgraphs C_1 and C_2 belong to different equivalence classes, we have $z_1 \in W_{x_k}(C_1) \cap W_{y_l}(C_2)$, $z_2 \in W_{x_k}(C_1) \cap W_{y_m}(C_2)$ for some vertices $x_k \in C_1$, $y_l, y_m \in C_2$. Assume that $N_{x_k}(C_2) = \{y_t\}$, $y_t \neq y_m$. Then $d(x_1, y_1) = d(x_2, y_2) = d(x_k, y_t)$. Since $d(y_m, x_k) = d(y_t, x_k) + 1$, then $y_m \in W(C_1)$. This contradicts d-convexity of the set $W_{x_k}(C_1)$, since $y_t, z_2 \in W_{x_k}(C_1)$ and $y_m \in \langle x_2, y_t\rangle$. Thus, for every edge e = (x, y) there exists a unique equivalence class $\bar{c}_i = \bar{c}(e)$ from Σ_0 whose subgraphs are comparable with e.

Let $\bar{c}_i \equiv \{W_1^i, \ldots, W_{a_i}^i\}$, $i = \overline{1, \ldots, n}$ and consider the following embedding of the graph G in the Hamming graph $H_{a_1 \ldots a_n}$. To the vertex x associate the tuple $\alpha(x) = (x_1, \ldots, x_n)$, where $x_i = k$ if $x \in W_k^i$. Since every family $\{W_1^i, \ldots, W_{a_i}^i\}$, $i = \overline{1, \ldots, n}$, is a partition of the set X, the i-coordinate of the vertex x is uniquely defined.

We will now show that the mapping α defined above is an isometric embedding of the graph G in $H_{a_1 \ldots a_n}$. Consider arbitrary vertices $x, y \in X$, $d(x, y) = m$ and let $\alpha(x) = (x_1, \ldots, x_n)$, $\alpha(y) = (y_1, \ldots, y_n)$; $(x = v_1, v_2, \ldots, v_m, v_{m+1} = y)$ is some shortest chain between x and y. Since every edge in this chain is comparable with subgraphs of a unique equivalence class from Σ_0, the tuples $\alpha(x)$ and $\alpha(y)$ differ at most in m places. For the edge $e_j = (v_j + v_{j+1})$, $j = \overline{1, \ldots, m}$, let $\bar{c}_{i_j} = \bar{c}(e_j)$. Let $W_{v_j}(e_j) = W_1^{i_j}$, $W_{v_{j+1}}(e_j) = W_2^{i_j}$. Since the sets $X \setminus W_1^{i_j}$, $X \setminus W_2^{i_j}$ are d-convex, then $v_1, \ldots, v_{j-1}, v_j \in W_1^{i_j}$, $v_{j+1}, \ldots, v_m, v_{m+1} \in W_2^{i_j}$. Therefore, the equivalence classes $\bar{c}_{i_1}, \ldots, \bar{c}_{i_m}$ are pairwise distinct. Since $x = v_1 \in W_1^{i_j}$, $y = v_{m+1} \in W_2^{i_j}$, the tuples $\alpha(x)$ and $\alpha(y)$ differ in m coordinates.

2 $\Rightarrow$ 1). By Lemma 2, it suffices to show that any block graph T is isometrically embeddable in a Hamming graph. In the graph T, a set is d-convex only if it induces a connected subgraph [15]. Let the edge e = (x, y) be contained in the block B. The set B is Chebyshev, and $W_z(B)$ induces a connected subgraph for any vertex $z \in B$. Then for the edge e,

the sets $W_x(e) = W_x(B)$, $W_y(e) = W_y(B)$, $W(e) = \bigcup\{W_z(B): z \neq x, y\}$ and their complements are d-convex. QED.

THEOREM 2. For a connected graph G = (X, U) the following conditions are equivalent:

1) G is isometrically embeddable in a hypercube;

2) G is isometrically embeddable in the direct product of trees;

3) any set $A \subset X$, $|A| \leqslant 5$, is isometrically embeddable in a hypercube;

4) G is bipartite and for every edge e = (x, y) the sets $W_x(e)$, $W_y(e)$ are d-convex [8].

Proof. The implications 1) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1) follow from the previous result. The implication 1)$\Rightarrow$2) follows from Lemma 2 and d-convexity of the sets $W_x(e)$, $W_y(e)$ for every edge e = (x, y) of the tree T.

Remark 1. The equivalence 1) $\Leftrightarrow$ 4) is the main result of Djokovic [8].

Remark 2. The isometric index $i(\mathcal{F}_1, \mathcal{F}_2)$ of the class of graphs $\mathcal{F}_2$ relative to the family of graphs $\mathcal{F}_1$ is the least number k such that the graph G = (X, U) from $\mathcal{F}_2$ is isometrically embeddable in some graph from $\mathcal{F}_1$ if and only if every set $A \subset X$, $|A| \leqslant k$ is isometrically embeddable in some graph from $\mathcal{F}_1$. If $\mathcal{F}_2$ is the collection of all connected graphs and $\mathcal{F}_1$ the family of all hypercubes or Hamming graphs, then $i(\mathcal{F}_1, \mathcal{F}_2) = 5$. Indeed, from Theorems 1 and 2 it follows that $i(\mathcal{F}_1, \mathcal{F}_2) \leqslant 5$, and the example of the graph $K_{2,3}$ proves that this inequality is exact.

Let us provide a more constructive description of isometric subgraphs of Hamming graphs. To this end, we introduce the notion of isometric expansion and contraction of graphs (note that related concepts are introduced in [2]).

Consider the graph $G_0 = (X_0, U_0)$, where $X_0 = \{x^1, \ldots, x^m\}$, and choose the sets $W_1^0, \ldots, W_n^0$ in this graph forming a cover of the set X_0 and satisfying the conditions 1)-3) for any i, $j = \overline{1, \ldots n}$:

1) $W_i^0 \cap W_j^0 \neq \varnothing$;

2) $\{(x^l, x^k) \in U_0 : x^l \in W_i^0 \setminus W_j^0,\ x^k \in W_j^0 \setminus W_i^0\} = \varnothing$;

3) the subgraphs induced by the sets $W_i^0, W_j^0, W_i^0 \cup W_j^0$ are isometric in the graph G_0.

To each vertex $x \in X_0$ associate the tuple $(i_1, \ldots, i_k)$ of all indexes i_j such that $x \in W_{i_j}^0$. The graph G = (X, U) is called an isometric expansion of the graph G_0 relative to the sets $W_1^0, \ldots, W_n^0$ if it is obtained from G_0 in the following way:

a) if the tuple $(i_1, \ldots, i_k)$ corresponds to the vertex x^j, $j = \overline{1, \ldots, m}$, then replace this vertex in G with pairwise adjacent vertices $x_{i_1}^j, \ldots, x_{i_k}^j$;

b) if the index i_j belongs to both tuples $(i_1', \ldots, i_{k_1}')$, $(i_1'', \ldots, i_{k_2}'')$ corresponding to the vertices $x^{j_1}, x^{j_2} \in X_0$, then in the graph G let $(x_{i_j}^{j_1}, x_{i_j}^{j_2}) \in U$.

If n = 2, then we assume that G is obtained from G_0 by binary isometric expansion.

The operation of isometric expansion of the graph G_0 to G may be considered as some many-valued mapping $\psi : X_0 \to X$, where the image $\psi(x^j)$ of the vertex x^j with the tuple $(i_1, \ldots, i_k)$ is the set $\{x_{i_1}^j, \ldots, x_{i_k}^j\}$.

Let W_i be the set of all vertices of G having the form x_i^j, $j = \overline{1, \ldots, m}$. The mapping $\varphi = \psi^{-1}$, which is the inverse of ψ, is called isometric contraction of the graph G to the graph G_0 relative to the sets $W_1, \ldots, W_n$.

THEOREM 3. The graph G = (X, U) is isometrically embeddable in a Hamming graph if and only if it is obtained from a one-vertex graph by a sequence of isometric expansions.

Proof. Let G be an isometric subgraph of the graph $H = H_{a_1} \times \ldots \times H_{a_n}$, where H has the least number of vertices among all Hamming graphs in which G is isometrically embeddable. We will show that G can be isometrically contracted to some graph G with fewer vertices which is isometrically embeddable in a Hamming graph. Consider the complete subgraph C of the graph G with maximum number of vertices. By the choice of H, C has the maximum number of vertices among the complete subgraphs of H. Without loss of generality, we may take $|C| = a_n$. Let $C = \{x_1, \ldots, x_{a_n}\}$. The graph H is representable in the form $H_0 \times H_{a_n}$, where $H_0 = H_{a_1} \times \ldots \times H_{a_{n-1}}$. It is easy to see that the sets $W_i = W_{x_i}(C)$ are contained in $H^{(i)} = H_0 \times \{i\}$, $i = \overline{1, \ldots, a_n}$. The set C is Chebyshev, and therefore the family $\{W_{x_i}(C) : x_i \in C\}$ partitions the set X into d-convex subsets. The vertices from $H^{(i)}$, $i=\overline{1,\ldots,a_n}$, will be denoted by $x_1^{(i)},\ldots,x_m^{(i)}$, where $m=a_1\cdot\ldots\cdot a_{n-1}$ so that the vertices $x_l^{(1)}\in H^{(1)},\ldots,x_l^{(a_n)}\in H^{(n)}$, $l=\overline{1,\ldots,m}$, induce a complete subgraph in H. The Hamming graph $H_0 = (V_0, U_0)$, where $V_0 = \{y_1,\ldots,y_m\}$, may be obtained from H by identifying the vertices $x_l^{(1)},\ldots,x_l^{(a_n)}$, $l=\overline{1,\ldots,m}$, with the vertex y_ℓ. This contradiction φ transforms the graph G into a subgraph G_0 of H_0. It is easy to see that G_0 is obtained from G by isometric contraction relative to the sets $W_1,\ldots,W_n$. Indeed, since the sets W_i, $W_j\cup W_k$ $i,j,k=\overline{1,\ldots,a_n}$, are d-convex in G, the sets $W_i^0=\bigcup\{\varphi(x):x\in W_i\}$, $W_j^0\cup W_k^0$, $i,j,k=\overline{1,\ldots,a_n}$, induce isometric subgraphs of G_0. Since C is a Chebyshev set, then $\bigcap_{i=1}^{a_n} W_i^0\neq\varnothing$, $\bigcup_{i=1}^{a_n} W_i^0=X_0$. The definition of identification implies that $\{(y_l,y_k)\in U_0: y_l\in W_i^0\setminus W_j^0, y_k\in W_j^0\setminus W_i^0\}=\varnothing$ for any indexes $i,j=\overline{1,\ldots,a_n}$. We will now show that G_0 is an isometric subgraph of the Hamming graph H_0. Let y_i, y_j be arbitrary vertices from G_0. If for some index ℓ the vertices $x_i^{(l)}$, $x_j^{(l)}$ belong to the set W_i, then $d_{G_0}(y_i,y_j)=d_G(x_i^{(l)},x_j^{(l)})=d_H(x_i^{(l)},x_j^{(l)})=d_{H_0}(y_i,y_j)$. Consider the case when $x_i^{(l)}\in W_l$, $x_j^{(l)}\notin W_l$, $x_i^{(k)}\in W_k$, $x_j^{(k)}\in W_k$. Since the set $W_l\cup W_k$ induces an isometric subgraph of G, then there exists a shortest chain from $x_i^{(l)}$ to $x_j^{(k)}$ which is entirely contained in $W_l\cup W_k$. On this chain, there are adjacent vertices $x_p^{(l)}$, $x_p^{(k)}$ such that $x_p^{(l)}\in W_l$, $x_p^{(k)}\in W_k$. We thus have $d_{G_0}(y_i,y_j)=d_G(x_i^{(l)},x_j^{(k)})-1=d_{H_0}(y_i,y_j)$, i.e., G_0 is an isometric subgraph of H_0.

Now assume that the graph G is obtained from G_0 by isometric expansion ψ relative to the sets $W_1^0,\ldots,W_m^0$. If G_0 is an isometric subgraph of $H_0=H_{a_1}\times\ldots\times H_{a_{n-1}}$, we can show that G is isometrically embeddable in the Hamming graph $H = H_0 \times H_m$. Consider an arbitrary vertex $x_i = \psi(x)\cap W_i$, where the vertex x corresponds to the tuple $(x^1,\ldots,x^{n-1})$. To the vertex x_i associate the n-vector $(x^1,\ldots,x^{n-1},i)$. We will now show that we have obtained an isometric embedding of G in H. If the vertices x_i, y_i belong to the set W_i, then the distance $d(x_i, y_i)$ is equal to d(x, y), i.e., it coincides with the number of different pairs of coordinates in the tuples $(x^1,\ldots,x^{n-1},i)$, $(y^1,\ldots,y^{n-1},i)$. Now let $x_i\in W_i$, $y_j\in W_j$. Since the set $W_i^0\cup W_j^0$ induces an isometric subgraph of G_0, we have the equalities $d(x_i,y_j)=d_{G_0}(x,y)+1=d_{H_0}(x,y)+1=d_H(x_i,y_j)$. The last equality follows from the fact that the Hamming distance between the tuples $(x^1,\ldots,x^{n-1},i)$, $(y^1,\ldots,y^{n-1},j)$ is $d_{H_0}(x, y) + 1$. QED.

A direct corollary of this result is the following constructive description of isometric subgraphs of hypercubes.

THEOREM 4. The graph G is isometrically embeddable in a hypercube if and only if it is obtained from a one-vertex graph by a sequence of binary isometric expansions.

Remark 3. Theorems 1 and 2 are true also for graphs with any (not necessarily finite) number of vertices [16].

LITERATURE CITED

1. P. M. Winkler, "Isometric embedding in product of complete graphs," Discrete Appl. Math., 7, No. 2, 221-225 (1984).
2. H.-M. Mulder, "The interval function of a graph," Math. Center Tracts, No. 132 (1983).
3. V. G. Boltyanskii and P. S. Soltan, Combinatorial Geometry of Various Classes of Convex Sets [in Russian], Shtiintsa, Kishinev (1978).
4. T. T. Arkhipova and I. V. Sergienko, "On formalization and solution of some problems of the computational process in data processing systems," Kibernetika, No. 5, 11-18 (1973).
5. L. P. Vlasov, "Approximative properties of sets in linear normed spaces," Usp. Mat. Nauk, 28, No. 8, 3-66 (1972).
6. R. L. Graham and H. O. Pollak, "On the addressing problem for loop switching," J. Bell. Syst. Tech., 50, No. 8, 2495-2519 (1971).
7. R. L. Graham and H. O. Pollak, "On embedding graphs in squashed cubes," Lect. Notes Math., 302, 99-110 (1972).
8. D. Ž. Djoković, "Distance-preserving subgraphs of hypercubes," J. Combin. Theory, 14, No. 3, 263-267 (1973).
9. P. S. Soltan, D. K. Zambitskii, and K. F. Prisakaru, Extremal Problems on Graphs and Algorithms for Their Solutions [in Russian], Shtiintsa, Kishinev (1973).
10. P. Assouad, "Embeddability of regular polytopes and honeycombs in hypercubes," The Geometric Vein: The Coxeter Festchrift, Springer, Berlin (1981), pp. 141-147.
11. R. L. Graham and P. M. Winkler, "On isometric embedding of graphs," Trans. Am. Math. Soc., 288, No. 2, 527-536 (1985).

12. S. V. Yushmanov, "Reconstructing a graph from some set of columns of its distance matrix," Mat. Zametki, 31, No. 4, 641-645 (1982).
13. F. Harary, R. Melter, and I. Tomescu, "Digital metrics: A graph-theoretical approach," Pattern Recogn. Lett., 2, No. 3, 159-165 (1984).
14. V. P. Soltan, An Introduction to Axiomatric Convexity Theory [in Russian], Shtiintsa, Kishinev (1984).
15. R. E. Jamison-Waldner, "Convexity and block graphs," Congr. Numer., No. 33, 129-142 (1981).
16. V. D. Chepoi, d-Convex Sets on Graphs [in Russian], Abstract of Thesis, Minsk (1987).