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A Helly theorem in weakly modular space

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Abstract

The d-convex sets in a metric space are those subsets which include the metric interval between any two of its elements. Weak modularity is a certain interval property for triples of points. The d-convexity of a discrete weakly modular space X coincides with the geodesic convexity of the graph formed by the two-point intervals in X. The Helly number of such a space X turns out to be the same as the clique number of the associated graph. This result thus entails a Helly theorem for quasi-median graphs, pseudo-modular graphs, and bridged graphs.

1. Introduction

The well-known theorem of Helly says that each finite family of convex sets in $\mathbb{R}^n$ has a nonempty intersection provided each subfamily of at most $n+1$ sets has nonempty intersection. This result and its relatives form a central theme in abstract convexity [25, 27]. The *Helly number* of an abstract convexity is the smallest number $h \geqslant 2$ such that every finite family of convex sets meeting h by h has a nonempty intersection. A lower bound on h (relevant in the case of discrete convexities) is the largest size ω of a clique, that is, a maximal set whose subsets are all convex. This number ω is then called the *clique number*. For the minimal path convexity of a graph Helly number and clique number are actually equal [20, 15]. The same is also true in finite convex geometries [19]. In the case of the geodesic convexity of graphs the Helly number h may well exceed the clique number ω (the 5-cycle being the smallest example), so that it is interesting to find particular classes where equality of those two numbers holds. Chordal graphs are of this kind as was shown by Chepoi [10]. This result was then generalized to dismantlable graphs [6] as well as to pseudo-modular graphs [6, 14].

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In this paper we will further extend this Helly-type theorem for the geodesic convexity (alias d-convexity) to larger classes of graphs and metric spaces. We propose a generalization of metric spaces that apparently represents the right setting for formulating our results. Then we investigate the relationship between the d-convexity of a metric space (or its generalization) and the geodesic convexity of the associated graph: we give a sufficient interval condition (weaker than the one presented by Bandelt [2]) that guarantees the coincidence of both convexities. The stronger requirement of weak modularity is introduced and briefly studied in the subsequent section. For metric spaces in which all (metric) triangles are equilateral the Helly number is shown to be bounded from above by what we call the simplex number of the space. This paves the way to the desired Helly theorem for weakly modular spaces, which is proved in the final section.

2. Discrete geometric interval spaces

Let X be any (not necessarily finite) set. For each pair u, v of points in X, let uv be a subset of X, called the *interval* between u and v. Then X is an *interval space* [27] if and only if

$$u \in uv \quad \text{and} \quad uv = vu$$

for all $u, v \in X$. Every interval space gives rise to a convexity: a subset A of X is *convex* if and only if $uv \subseteq A$ for all $u, v \in A$; note that the intervals are in general not convex. An interval uv is called an *edge* if $u \neq v$ and $uv = \{u, v\}$; the edges then form the *graph* of the interval space X. In order to ensure that the graph of the space X is connected, some extra conditions are necessary. The interval space X is said to be *geometric* if it satisfies the following three conditions ($u, v, w, x \in X$) [9, 28]:

$$uu = \{u\},$$

$$w \in uv \text{ implies } uw \subseteq uv,$$

$$v, w \in ux \text{ and } v \in uw \text{ implies } w \in vx;$$

For each point u one defines the *base-point relation at* u as follows:

$$x \leqslant_u y \text{ if and only if } x \in uy.$$

Our first lemma summarizes some equivalent descriptions of geometric interval spaces [27, Section I.5.2].

Lemma 1. *For an interval space X the following conditions are equivalent*:

(i) *X is geometric*;

(ii) *$w \in ux$ and $x \in uw$ implies $w = x$,*
$v \in uw$ and *$w \in ux$ implies $v \in ux$ and $w \in vx$ ($u, v, w, x \in X$)*;

(iii) *for each point u the base-point relation* $\leqslant_u$ *is a partial order such that for any* $v \leqslant_u w$

$$vw = \{x : v \leqslant_u x \leqslant_u w\}.$$

Note that a geometric interval space is exactly a ternary space [18] satisfying the above property that $u \in uv$ for all points u, v; see also her paper for references to much earlier, related material on 'betweenness'.

A particular instance of geometric interval space is, of course, any metric space X: the intervals are the metric intervals

$$uv = \{x : \delta(u, x) + \delta(x, v) = \delta(u, v)\},$$

where δ is the metric of X. In general, a geometric interval space cannot be turned into a metric space such that the given intervals coincide with the metric intervals. To give an example, consider the set $X = \{u, v, x_1, x_2, z_1, z_2\}$ equipped with eight edges ux_i, vx_i, uz_i, vz_i $(i = 1, 2)$ and further intervals

$$uv = X,$$

$$x_iz_j = \{u, v, x_i, z_j\} \text{ for } i, j \in \{1, 2\},$$

$$x_1x_2 = \{u, v, x_1, x_2\},$$

$$z_1z_2 = \{u, z_1, z_2\}.$$

It is easily seen that X is a geometric interval space such that either subspace $X - \{z_i\}$ is isomorphic to the interval space of the complete bipartite graph $K_{2,3}$. Hence, if the intervals were derived from some metric δ, then all edges would receive the same length (cf. Section 4 below), and therefore

$$\delta(z_1, u) + \delta(u, z_2) = \delta(z_1, v) + \delta(v, z_2),$$

thus contradicting the definition of z_1z_2.

A *chain* R in an interval space X is a subset which can be linearly ordered by $\leqslant$ such that for $u, v, w \in R$ one has $v \in uw$ if and only if $u \leqslant v \leqslant w$. If R admits a least element a and a largest element b, then R is called *bounded*. Now, X is said to be *discrete* if all bounded chains in X are finite.

Lemma 2. *Let X be a discrete geometric interval space. Then every maximal chain R between two points u and v of X is an induced path in the graph of X. In particular, the graph of X is connected.*

Proof. Since X is discrete, R must be finite. If z covers x in the linear order of R, then for every point y of X with $y \in xz$, the extension $R \cup \{y\}$ is again a chain because X is geometric [18, Lemma 1.1]. Therefore $y \in \{x, z\}$, which shows that R is a path in the graph of X. Clearly R is an induced path (i.e., there are no additional edges). $\square$

The graph of a discrete geometric interval space X can be regarded as a metric space, where the metric d accounts for the lengths of shortest paths in the graph. The corresponding intervals

$$I(u, v) = \{x : x \text{ is on a shortest path between } u \text{ and } v\}$$

have to be distinguished from the intervals uv of the given interval space (at least, when they contain more than two points). Now, call an interval space X *graphic* if the equality $uv = I(u, v)$ holds for all points u, v of the space. So, the graphic interval spaces are exactly the interval spaces obtained from connected graphs. A number of metric properties of graphs can be formulated in terms of intervals [21]. Graphic interval spaces are necessarily discrete and geometric; a sufficient condition is presented in the next section.

3. The triangle condition

An interval space X is said to satisfy the *triangle condition* if for any three points u, v, w in X with

$$uv \cap uw = \{u\}, \qquad uv \cap vw = \{v\}, \qquad uw \cap vw = \{w\},$$

the intervals uv, uw, vw are edges whenever at least one of them is an edge.

Theorem 1. *A discrete geometric interval space X satisfying the triangle condition is graphic.*

Proof. We have to show $uv = I(u, v)$. Proceed by induction on $n = d(u, v)$. For $n \leqslant 1$ there is nothing to show because $uv = \{u, v\}$. Now, let $n \geqslant 2$, and assume that $xy = I(x, y)$ whenever $d(x, y) \leqslant n - 1$. Let u, v be two points at distance n:

$$d(u, v) = n \geqslant 2.$$

Assertion 1. $I(u, v) \subseteq uv$.

Suppose this fails. Then there exists a (shortest) path P joining u and v with n edges which is not a chain in the space X. Let w be the point on P adjacent to v. Then the subpath of P from u to w lies in $I(u, w) = uw$ and hence is a chain, by virtue of the initial assumption. Therefore w does not belong to uv, for otherwise, P were a chain. Now, $v \in uw = I(u, w)$ is impossible as $w \in I(u, v)$. Hence there exists a point $x \in uv \cap uw$ with $vx \cap wx = \{x\}$ because the space is discrete. Since vw is an edge outside uv and uw, we get $vw \cap vx = \{v\}$ and $vw \cap wx = \{w\}$. Hence, by the triangle condition, vx and wx are edges. Then $d(u, x) = n - 2$ as $x \in uw = I(u, w)$. This yields $d(u, v) \leqslant 1 + d(u, x) = n - 1$, a contradiction.

We conclude that $I(u, v) \subseteq uv$. So, $I(x, y) \subseteq xy$ whenever $d(x, y) \leqslant n$.

Assertion 2. $uv \subseteq I(u, v)$.

Suppose the contrary: then there is a maximal chain R between v and u with $m > n$ edges. Let $x_0 = v$, $x_1, \dots, x_{m-1}$, $x_m = u$ be the points on R, so that $x_j \in x_i x_r$ for $i < j < k$. Pick any shortest path Q joining u and v in the graph of X, where $w_0 = u$, $w_1, \dots, w_{n-1}$, $w_n = v$ are the points of Q such that each w_i is adjacent to w_{i-1}. From Assertion 1 we know that Q is also a chain. Put $x = x_1$ and $w = w_1$. Observe that $d(u, x) \geqslant n$, for otherwise, we would get $ux = I(u, x)$, so that R would be a shortest path.

Claim 1. $v \in wx$.

Suppose this fails, but $x \in vw$ holds. Then, as $vw = I(v, w)$, we get $d(w, x) = n - 2$, thus giving $d(u, x) \leqslant n - 1$, a contradiction. Therefore we may suppose that the edge vx is outside vw and wx. Consequently, since X is discrete and satisfies the triangle condition, we can find a common neighbor y of v and x belonging to $vw \cap wx$. Then, as $vw = I(v, w)$, we get $d(w, y) = n - 2$, whence

$$n \leqslant d(u, x) \leqslant d(u, w) + d(w, x) \leqslant n.$$

It follows that $w \in I(u, x) \subseteq ux$ by virtue of Assertion 1. Now, since $x \in uv$ and X is geometric, we obtain $x \in vw$, a contradiction. This establishes Claim 1.

Note that $d(w, x) = n$ holds, for otherwise, we would obtain $d(w, x) \leqslant n - 1$ and hence v would not belong to $I(w, x)$, thus conflicting with Claim 1.

Claim 2. $u \in wx$.

Suppose this fails. If $w \in ux$ were true, then from $x \in uv$ we would infer $x \in vw = I(v, w)$ because X is geometric, thus yielding

$$d(u, x) \leqslant d(u, w) + d(w, x) = n - 1,$$

which is impossible. Therefore we may suppose that the edge uw is neither in ux nor in wx. Similarly, as in Claim 1, we get a point $y \in ux \cap wx$ which is adjacent to u and w. Since $d(v, w) = n - 1$ and $d(u, v) = n$, we must have

$$n - 1 \leqslant d(v, y) \leqslant n.$$

If $d(v, y) = n - 1$, then $vy = I(v, y)$. Since X is geometric, we infer from $x \in uv$ and $y \in ux$ that $x \in vy$, whence $d(x, y) = n - 2$ and then $d(u, x) \leqslant n - 1$, again being impossible. So, we can assume $d(v, y) = n$. It follows that $w \in I(v, y) \subseteq vy$. In view of $y \in ux \subseteq uv$, this gives $y \in uw$ since X is geometric. This, however, is absurd by the choice of y, thus establishing Claim 2.

Summarizing, we have $w, x \in uv$ as well as $u, v \in wx$ such that $d(u, v) = d(w, x) = n$. Then the path Q_1 composed by the subpath of Q from w to v and the edge vx is a shortest path between w and x. Further, the subpath of R from x to u and the edge uw constitute a maximal chain R_1 between x and w with m edges. Then, by letting Q_1 play the role of Q and R_1 the role of R, we get a shortest path Q_2 between w_2 and x_2 as well as a maximal chain R_2 between x_2 and w_2. And so on, until we arrive at a shortest path Q_n from $w_n = v$ to x_n, being a proper subpath of the chain R, as well as a maximal chain R_n from x_n to w_n. It follows $u = x_m \in w_n x_n = v x_n$. Since X is geometric and $x_n \in uv$, we conclude that $x_m = u = x_n$, contrary to $m > n$. This final contradiction proves Assertion 2 and thus completes the proof of Theorem 1. □

This theorem extends Lemma 1 of Bandelt [2], which was formulated for finite metric spaces under the stronger requirement that for every edge vw and any point u either $v \in uw$ or $w \in uv$ hold. A still stronger condition is *modularity*, viz., $uv \cap uw \cap vw$ is nonempty for all points u, v, w; see Bandelt et al. [9]. In the case that all these intersections are singletons one arrives at *median spaces* [2, 24, 27].

The triangle condition also holds for a discrete geometric interval space X in which every induced path is a chain. To see this, suppose the triangle condition fails for a triple u, v, w so that vw is an edge but uw is not. Choose any maximal chain R between u and v. Let x be that point on R adjacent to w which is closest to u. Since uw is not an edge, x is different from u. Then the subpath of R from u to x and the edge xw give an induced path between u and w. By hypothesis, this yields $x \in uv \cap uw$, contrary to $uv \cap uw = \{u\}$. This proves the claim. Graphs in which all induced paths are shortest paths are called *distance-hereditary*; cf. [5]. Hence the following observation obtains (in view of Theorem 1).

Remark 1. A discrete geometric interval space X is the interval space of a distance-hereditary graph if and only if every induced path in the graph of X is a chain in X.

4. Weak modularity

The triangle condition for an interval space X is weaker than modularity (requiring that each intersection $uv \cap uw \cap vw$ be nonempty). There is yet another generalization of modularity: we say that X enjoys *interval-constrained* modularity if

$$v, w \in ux \text{ and } x \in vw \text{ implies that } uv \cap uw \cap vw \text{ is nonempty,}$$

for all points u, v, w, x of X. A related property (also implied by modularity) is the following: X is said to satisfy the *quadrangle condition* if $v, w \in ux$ and $x \in vw$ such that vx and wx are edges implies that $uv \cap uw \cap vw$ contains a point y such that vy and wy are edges. For graphical interval spaces the quadrangle condition amounts to interval-constrained modularity:

Remark 2. A connected graph X enjoys interval-constrained modularity exactly when X satisfies the quadrangle condition.

The proof of this observation is easy: evidently, the quadrangle condition follows from interval-constrained modularity is this case. As to the converse, proceed by induction on $n = d(v, w)$, where $v, w \in I(u, x)$ and $x \in I(v, w)$. Suppose that $I(w, x)$ is not an edge. Let x' be a neighbor of x in $I(w, x)$. Then, by virtue of the induction hypothesis, there exists $v' \in I(u, v) \cap I(u, x') \cap I(v, x')$ such that $x' \in I(v', w)$. Applying the hypothesis now to u, v', w, the required point is obtained.

Call an interval space X *weakly modular* if it satisfies interval-constrained modularity as well as the triangle condition. For example, every median algebra gives rise to a weakly modular space; cf. [3, 9, 27]. In view of Theorem 1 and Remark 2, a discrete geometric interval space is weakly modular if and only if it satisfies the triangle and quadrangle conditions. Weakly modular graphs were previously studied by Bandelt and Mulder [8] and Chepoi [11, 12]. Particular subclasses are formed by the pseudo-modular graphs [4], quasi-median graphs [21, 29, 30, 13], and bridged graphs [1, 17, 26]. Among the pseudo-modular graphs one finds all pseudo-median graphs [7], absolute retracts (cf. [23]), and distance-hereditary graphs.

In the case of graphs, weak modularity can be expressed by a single condition; see Theorem 2 of Chepoi [11]: a connected graph X is weakly modular if and only if for any three points u, v, w in X with

$$I(u,v) \cap I(u,w) = \{u\}, \qquad I(u,v) \cap I(v,w) = \{v\}, \qquad I(u,w) \cap I(v,w) = \{w\}$$

all points on every shortest path from v to w have the same distance to u. This result can also be deduced from [8]: we know from the proof of their Lemma 1 that in the presence of weak modularity every shortest path $v = v_0, v_1, \ldots, v_k = w$ can be transformed into a path $v = w_0, w_1, \ldots, w_k = w$ such that there are indices $h \leqslant j$ with

$$w_h \in I(u, v), \qquad w_j \in I(u, w),$$

$$d(u, w_i) = d(u, w_j) \quad \text{for all } i \text{ with } h \leqslant i \leqslant j.$$

By the choice of u, v, w it follows that $h = 0$ and $j = k$.

We conclude this section by having a brief look at weakly modular metric spaces. The metric δ of any discrete metric space X restricts to a positive weight function on the edge-set of X. When does, conversely, a weight function λ on the edge-set of a graph X give rise to a graphic metric space? If X is a weakly modular graph, then the feasible functions λ can be described conveniently by only checking the triangles and 4-cycles (with at most one chord). The next result thus generalizes Lemma 2 of Bandelt [2] and is proved similarly.

Proposition 1. *Let X be a weakly modular graph, and let λ be a positive weight function on the edge-set of X. Then the resulting metric space is graphic, that is, it has the same* *intervals as the given graph X if and only if λ fulfills the following two conditions*:

$$\lambda(uv) < \lambda(uw) + \lambda(wv) \quad \textit{for every triangle } \{u, v, w\};$$

$$\lambda(ux) + \lambda(xv) = \lambda(uy) + \lambda(yv) \quad \textit{for any two common neighbors } x, y \textit{ of two nonadjacent points } u \textit{ and } v.$$

Proof. Necessity is clear. As to sufficiency, let δ be the metric induced by λ. More explicitly, for a path P from u_0 to u_n with n edges u_iu_{i+1}, *let* $\lambda(P)$ be the sum of all $\lambda(u_iu_{i+1})$ for $i = 0, \ldots, n-1$. Then $\delta(u_0, u_n)$ is the minimum of all $\lambda(Q)$, where Q is a path from u_0 to u_n.

Assertion 1. *If P and Q are two shortest paths between u and v in the graph, then $\lambda(P) = \lambda(Q)$.*

To verify this, proceed by induction on the number $n = d(u, v)$ of edges on either path. Let x and y be the points on P and Q, respectively, which are adjacent to v. If $x = y$, then $\lambda(P) = \lambda(Q)$ is immediate from the induction hypothesis. Otherwise, by weak modularity, there is a common neighbor z of x and y with $d(u, z) = n - 2$. Applying the induction hypothesis to the respective subpath P' and Q' of P and Q between u and the corresponding neighbor of v, we get

$$\begin{aligned}\lambda(P) &= \lambda(vx) + \lambda(P') = \lambda(vx) + \lambda(xz) + \lambda(R)\\ &= \lambda(vy) + \lambda(yz) + \lambda(R) = \lambda(vy) + \lambda(Q') = \lambda(Q),\end{aligned}$$

where R is any shortest path between u and z.

Assertion 2. $uv \subseteq I(u, v)$.

Suppose by way of contradiction that there is a maximal chain R between u and v which has $m > n = d(u, v)$ edges. Then $\lambda(R) = \delta(u, v)$. Assume that m is as small as possible. Let w be the neighbor of v on R, and let R' be the subpath of R from u to w. Clearly, $n - 1 \leqslant d(u, w) \leqslant n + 1$.

Case 1: $d(u, w) = n - 1$. Then R' has $m - 1 > d(u, w)$ edges, contrary to the minimality assumption.

Case 2: $d(u, w) = n$. Since X is weakly modular, there exists a common neighbor x of v and w with $d(u, x) = n - 1$. By the minimality assumptions, R' must have exactly n edges, so that R' is a shortest path. Then, according to the first assertion, $\lambda(R') = \lambda(Q) + \lambda(xw)$ for any shortest path Q from u to x. Hence

$$\lambda(R) = \lambda(Q) + \lambda(xw) + \lambda(wv) > \lambda(Q) + \lambda(vx)$$

by the first condition of the proposition. This, however, contradicts $\lambda(R) = \delta(u, v)$.

Case 3: $d(u, w) = n + 1$. Then, by the minimality assumption, R' has exactly $n + 1$ edges and hence is a shortest path between u and w. Therefore (by virtue of Assertion 1)

$$\lambda(R) = \lambda(R') + \lambda(vw) = \lambda(P) + 2\lambda(vw) > \lambda(P)$$

for any shortest path P between u and v, thus yielding a final contradiction. □

5. Metric spaces with equilateral triangles

In what follows X is a discrete geometric interval space (unless stated otherwise). A *triangle* in X consists of three distinct points u, v, w such that

$$uv \cap uw = \{u\}, \qquad uv \cap vw = \{v\}, \qquad uw \cap vw = \{w\}.$$

If, in addition, all three *sides* uv, vw, wu of this triangle are edges (so that u, v, w constitute a triangle in the graph of X), then we briefly say that $\{u, v, w\}$ is a *graphic triangle* in X. So, for instance, the triangle condition requires that a triangle in the given space be a graphic triangle whenever at least one of its sides is an edge. More generally, a *simplex* S in the space X is a nonempty subset of X such that any three distinct points in S form a triangle in X, and for any four distinct points u, v, w, x in S the intervals uv and wx are disjoint.

The *simplex number* $\sigma(X)$ of the space X is the maximum cardinality of finite simplices in X, or else it is infinite. The simplex number of the graph of X is called the *clique number* $\omega(X)$ in order to avoid confusion. The *Hadwiger number* $\eta(X)$ of the space is defined as the Hadwiger number of the associated graph, viz., it is either infinite or the largest number k for which there exists a partition of X into k connected subsets $A_1, \ldots, A_k$ such that for any two distinct indices i and j there is at least one edge uv between A_i and A_j, that is, $u \in A_i$ and $v \in A_j$.

Recall that a subset B of X is *convex* if B includes the interval between any two of its points. The *convex hull* $\mathrm{conv}(A)$ of $A \subseteq X$ is the smallest convex set containing A. A nonempty finite subset A of X is called *Helly independent* if

$$\bigcap_{a \in A} \mathrm{conv}(A - \{a\}) = \emptyset.$$

Then the *Helly number* $h(X)$ of X is the largest number k for which there is a Helly independent k-set A in X or is said to be infinite otherwise. We record the obvious relationship between the numbers $\omega(X)$, $\eta(X)$, and $h(X)$ in the following lemma.

Lemma 3. *For a discrete geometric interval space* X,

$$\omega(X) \leqslant h(X) \leqslant \eta(X) \text{ and } \omega(X) \leqslant \sigma(X) \leqslant \eta(X).$$

Proof. The inequalities $\omega(X) \leqslant h(X)$ and $\omega(X) \leqslant \sigma(X)$ are evident. Since all convex sets in X are connected in the graph of X, the inequality $h(X) \leqslant \eta(X)$ follows from

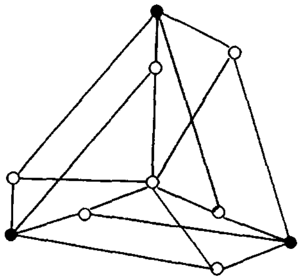

Fig. 1. $2 = h(X) < \sigma(X) = 3$.

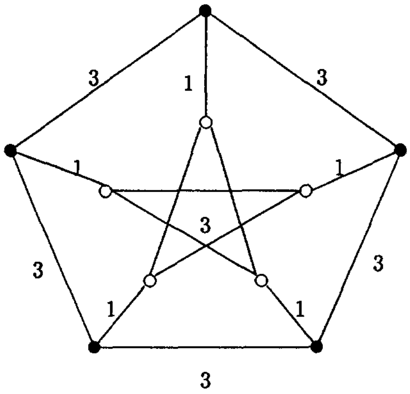

Fig. 2. $4 = \sigma(X) < h(X) = 5$.

Théorème 2.2 of Duchet and Meyniel [16]. The final inequality $\sigma(X) \leqslant \eta(X)$ is also easily verified: for any simplex $\{x_1, \ldots, x_k\}$ in X, each set

$$A_i = \bigcup_{j \neq i} (x_i x_j - \{x_j\})$$

is connected and contains x_i as well as a neighbor of each $x_j (j \neq i)$. □

Observe that no further inequalities between the four numbers hold in general. Indeed, the 4-cycle has Hadwiger number 3, but its Helly and simplex numbers equal 2. The 6-cycle has clique number 2, but its Helly and simplex numbers equal 3. Further, the graph X of Fig. 1 gives $h(X) = 2$ and $\sigma(X) = 3$ because the singletons and edges are the only nonempty convex sets, and the three points of degree 4 (shaded in the figure) form the unique triangle in the interval space X. Finally, the metric space X displayed in Fig. 2 confirms that the Helly number may exceed the simplex number. In fact, the five shaded points forming the outer 5-cycle are Helly independent by the particular choice of edge lengths. Clearly, as the maximum degree of the graph is 3 and there are only 10 points, the Hadwiger number cannot exceed 5, whence $h(X) = 5$.

The simplex number is at most the maximum degree plus 1. Actually, $\sigma(X) = 4$ is achieved by selecting any two edges u_1u_2 and v_1v_2 of length 3 such that no u_i is adjacent to any v_j.

This example with $h(X) = 5$ is in a way minimal. For, if $h(X) = 3$, then $\sigma(X) \geqslant 3$ obtains since a Helly independent set $\{u_1, u_2, u_3\}$ can be transformed into a triangle $\{v_1, v_2, v_3\}$ as follows: first substitute u_1 by a point v_1 in $u_1u_2 \cap u_1u_3$ such that $v_1u_2 \cap v_1u_3 = \{v_1\}$. Then, in a similar way, replace u_2 and finally u_3. Next assume that $h(X) = 4$ and X is a metric space with $\delta(u, v) \geqslant r$ for all distinct points u, v and some suitably chosen $r > 0$. Given a Helly independent set $\{u_1, u_2, u_3, u_4\}$, suppose there exists a point u'_1 in $u_1u_2 \cap u_1u_3$ different from u_1. Then substituting u_1 by u'_1 and letting $u'_i = u_i$ for $i \geqslant 2$ yields

$$\sum_{i<j} \delta(u'_i, u'_j) \leqslant \sum_{i<j} \delta(u_i, u_j) - r.$$

Continuing this way we eventually arrive at a Helly independent set $\{v_1, v_2, v_3, v_4\}$ (in fewer than $\sum_{i<j} \delta(u_i, u_j)/r$ steps) such that each triple is a triangle. Hence as $\{v_1, v_2, v_3, v_4\}$ is Helly independent it is a simplex, thus yielding $\sigma(X) \geqslant 4$.

The preceding strategy of 'shrinking' a Helly independent set in order to obtain a simplex works in the case $h(X) > 4$ under the additional assumption that all triangles in the space be equilateral. A triangle $\{u, v, w\}$ in X is called *equilateral* if its three sides have the same length, viz., $\delta(u, v) = \delta(u, w) = \delta(v, w)$. Note that a finite metric space the triangles of which are all equilateral need not be graphic: consider the 4-point space $\{u, v, w, x\}$ where u, v, w are pairwise at distance 2 and x has distance 1 to v and w but distance 3 to u.

Lemma 4. *Let X be a discrete metric space in which all triangles are equilateral. Then, for every convex set A and any point x outside A, a point u of A is at minimum distance to x if and only if $A \cap ux = \{u\}$. Moreover, every interval wx with $w \in A$ contains some point of A at minimum distance to x.*

Proof. Since X is discrete, wx contains some point $u \in A$ with $A \cap ux = \{u\}$. For any other point $v \in A$ with $A \cap vx = \{v\}$ we can find $y \in ux \cap vx$ such that $uy \cap vy = \{y\}$. Then, by the choice of u and v, the three points u, v, y form a triangle in X because A is convex. This triangle is equilateral by hypothesis, whence $\delta(u, x) = \delta(v, x)$. □

Lemma 5. *Let A be a Helly independent subset of a geometric interval space X. If $x \in uv \cap \mathrm{conv}(A - \{u\})$ for some points $u, v \in A$, then $B = (A - \{v\}) \cup \{x\}$ is Helly independent. In particular, B is Helly independent when x is chosen from $uv \cap vw$ for distinct $u, v, w \in A$.*

Proof. If $y \in A - \{u, v\}$, then $B - \{y\} \subseteq \mathrm{conv}(A - \{y\})$ since $x \in uv \subseteq \mathrm{conv}(A - \{y\})$. Further, $B - \{u\} \subseteq \mathrm{conv}(A - \{u\})$ because $x \in \mathrm{conv}(A - \{u\})$. Trivially, $B - \{x\} \subseteq$

$A - \{v\}$. Therefore

$$\bigcap_{y \in B} \operatorname{conv}(B - \{y\}) \subseteq \bigcap_{a \in A} \operatorname{conv}(A - \{a\}) = \emptyset,$$

and so B is Helly independent.

As to the second assertion, we claim that either $vw \subseteq A - \{u\}$ or $uv \subseteq A - \{w\}$, so that the first assertion applies in either case. Suppose that vw is not included in $A - \{u\}$, that is, $u \in vw$. Then as X is geometric we get $w \notin uv$, as required. □

Proposition 2. *Let X be a metric space such that all triangles in X are equilateral and $\delta(u, v) \geqslant r > 0$ holds for all distinct points u, v of X. Then the convex hull of every Helly independent finite set in X includes a Helly independent simplex of the same cardinality, thus yielding $h(X) \leqslant \sigma(X)$.*

Proof. For a finite subset Y of X and a point $x \in Y$ we use the shorthand

$$\Delta(x, Y) = \sum_{y \in Y} \delta(x, y) \quad \text{and} \quad \Delta(Y) = \min_{z \in Y} \Delta(z, Y).$$

Let A be a Helly independent finite set in X. Pick $u \in A$ with $\Delta(u, A) = \Delta(A)$. Suppose that some $v \in A - \{u\}$ is not at minimum distance to u among the points from $\operatorname{conv}(A - \{u\})$. Then, according to Lemma 4, the interval uv contains such a point x (necessarily different from v) at minimum distance to u. From Lemma 5 we infer that $B = (A - \{v\}) \cup \{x\}$ is a Helly independent set (having the same cardinality as A). Moreover,

$$\Delta(B) \leqslant \Delta(u, B) = \Delta(u, A) - \delta(v, x) \leqslant \Delta(u, A) - r.$$

Now, applying this argument to B instead of A and continuing, after at most $\Delta(u, A)/r$ steps one eventually arrives at a Helly independent set S of the same cardinality as A such that for each $z \in S$ with $\Delta(z, S) = \Delta(S)$ all points of $S - \{z\}$ are at minimum distance to z among the points of $\operatorname{conv}(S - \{z\})$. For any two distinct points $v, w \in S - \{z\}$ we obtain

$$vw \cap vz = \{v\} \quad \text{and} \quad vw \cap wz = \{w\}$$

since S is Helly independent (so that $z \notin vw$) and v, w are points of $\operatorname{conv}(S - \{z\})$ being at minimum distance to z. Choose any point y in $vz \cap wz$ with $vy \cap wy = \{y\}$. Then $\{v, w, y\}$ is a triangle, which must be equilateral by hypothesis. Hence

$$\delta(v, w) = \delta(w, y) = \delta(w, z) - \delta(y, z)$$

and therefore

$$\Delta(v, S) \leqslant \Delta(z, S) - \delta(y, z).$$

By the choice of z we have $\Delta(z, S) = \Delta(S) \leqslant \Delta(v, S)$, thus yielding $y = z$. We conclude that $\Delta(v, S) = \Delta(S)$ for all $v \in S$. In particular, every triple in S constitutes a triangle. Finally, as S is Helly independent, vw and ux are disjoint for all distinct points $u, v, w, x \in S$. This proves that S is a simplex. □

Note that the preceding proposition applies to weakly modular graphs in view of Theorem 2 of Chepoi [11]. The same result then also holds for discrete weakly modular spaces (because they are graphic), although triangles in a weakly modular metric space need not be equilateral (cf. the subsequent remark). On the other hand, the triangle condition alone does not suffice to guarantee the inequality $h(X) \leqslant \sigma(X)$. Indeed, turn the metric space depicted in Fig. 2 into a bipartite graph by substituting each edge of length j by a path with $2j$ new edges.

Remark 3. All triangles in a discrete weakly modular metric space X are equilateral if and only if all graph triangles are such.

Proof. Necessity is trivial. As to the converse, let $\{u, v, w\}$ be a triangle in X. Then, by Theorem 2 of [11], every point in vw has the same graph distance to u. For each edge xy with $x, y \in vw$ we can find some point $z \in ux \cap uy$ such that $\{x, y, z\}$ is a graphic triangle. It follows

$$\delta(u, x) = \delta(u, z) + \delta(z, x) = \delta(u, z) + \delta(z, y) = \delta(u, y),$$

as required. □

6. A Helly theorem

The above Proposition 2 already entails a certain Helly theorem as a particular case. Call an interval space X *pseudo-modular* if there are no triangles in X other than graphic triangles. Consequently, all simplices in X are complete subgraphs, thus giving $\omega(X) = \sigma(X)$. A discrete geometric pseudo-modular space X is graphic by Theorem 1, and its graph is pseudo-modular in the sense of Bandelt and Mulder [4]. Now, the graph of X is weakly modular (see Proposition 4 of [4]), and so the equality $h(X) = \sigma(X)$ follows from Proposition 2.

We extend the preceding argument by first showing that weak modularity suffices to ensure $\omega(X) = \sigma(X)$ and then applying Proposition 2.

Proposition 3. *Let X be a discrete geometric weakly modular space. Then the convex hull of a Helly independent finite simplex S in X includes a graphic simplex R* (i.e., *a complete subgraph of the graph of X*) *having the same cardinality.*

Proof. Recall that X is graphic, so that X can be regarded as a weakly modular graph. If $S = \{u, v, w\}$, then w together with any one of its neighbors in $I(v, w)$ lie on a (graphic) triangle by virtue of the triangle condition and Theorem 2 of [11]. So, let $|S| \geqslant 4$. Suppose that S is not yet a graphic simplex although it is chosen so that the (graph) distance $d(u,v)$ between any two points u and v in S is as small as possible. Then pick any neighbor x of v in $I(u,v)$ for some $u \neq v$ in S. According to Theorem 2 of [11], all points in $S - \{u,v\}$ have distance $d(u,v)$ to v and x. Hence, by the triangle condition, there exists a neighbor y' of v and x in $I(v,y) \cap I(x,y)$ for each $y \in S - \{u,v\}$. Letting $u' = x$ and $v' = v$, we then arrive at a new set $S' = \{z' : z \in S\}$ satisfying $|S'| = |S|$ (because S is a simplex). Suppose that S' is not a graphic simplex, say: $y' \neq z'$ in $S' - \{v\}$ are not adjacent. Then $x \in I(u,v) \cap \text{conv}(S - \{u\})$ because $x \in I(y', z')$, and so Lemma 5 provides us with a Helly independent set $(S - \{v\}) \cup \{x\}$ in which the minimum distance equals $d(u,v) - 1$. When applied to this set the shrinking process described in the proof of Proposition 2 returns a Helly independent simplex with minimum distance no larger than $d(u,v) - 1$. This, however, violates the initial assumption on S, thus completing the proof. □

Finally, we obtain the desired Helly theorem by combining Propositions 2 and 3:

Theorem 2. *The Helly and clique numbers of a discrete geometric weakly modular space X are equal: $h(X) = \omega(X)$.*

In particular, $h(X) = \omega(X)$ holds in a weakly modular graph X. This result can be combined with the corresponding Helly theorem for dismantlable graphs as follows. We say that a graph X can be *dismantled* to an induced subgraph Y of X if $X - Y = \{x_1, \ldots, x_m\}$ such that for each $i = 1, \ldots, m$ the point x_i and its neighbors in the graph $X - \{x_1, \ldots, x_{i-1}\}$ are all adjacent to some other point x_i' from $X - \{x_1, \ldots, x_i\}$. Graphs that can be dismantled to the singleton graph are called *dismantlable* or copwin [1, 6, 22, 23]. Then the inductive proof of Theorem 1 in the latter paper allows us to derive the following result from the above Theorem 2.

Corollary. *If a graph X can be dismantled to a weakly modular graph, then $h(X) = \omega(X)$ holds.*

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