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L_1–EMBEDDABILITY OF RECTILINEAR POLYGONS WITH HOLES

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The following result is established. Let P be a rectilinear polygon whose all holes are rectangles. If there are no maximal cut segments of P whose end–points lie on the boundary of different holes then P is L_1–embeddable.

In this note we investigate the L_1–embeddability of rectilinear polygons with holes endowed with the rectilinear metric. A metric space (X, d) is said to be *L_1–embeddable* if there is a measurable space $(\Omega, \mathcal{A})$, a nonnegative measure μ on it and an application λ of X into the set of measurable functions F (i.e. with $\|f\|_1 = \int_\Omega |f(w)|\mu(dw) < \infty$) such that

$$d(x, y) = \|\lambda(x) - \lambda(y)\|_1$$

for all $x, y \in X$ [1,2,5]. When Ω is a set of cardinality n and $\mathcal{A}$ is the collection of subsets of Ω and μ is the cardinality measure, i.e. $\mu(A) = |A|$ for $A \subseteq \Omega$, then $L_1(\Omega, \mathcal{A})$ is called a *n–dimensional hypercube.* A finite metric space (X, d) is *hypercube embeddable (h–embeddable,* for short) if there exist binary vectors $\gamma_1, \ldots, \gamma_m \in \{0,1\}^n$ such that $d(x_i, x_j) = \|\gamma_i - \gamma_j\|_1$ for any $x_i, x_j \in X$. Problems concerning L_1–embeddable and h–embeddable metric spaces have a long history, for a survey see [1,2,5].

The *interval* $I(u, v)$ between two points u, v of a metric space (X, d) consists of all points z between u and v, that is,

$$I(u, v) = \{x \in X : d(u, v) = d(u, x) + d(x, v)\}.$$

A particular instance of a metric space is any connected graph $G = (V, E)$ endowed with the standard shortest–path distance (the *distance* between vertices u and v of G is the

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number of edges in a shortest path from u to v). A subset S of X is *convex* if $I(u,v) \subseteq S$ for any points $u, v \in S$.

Below we present only those results on L_1–embeddable and h–embeddable spaces and graphs which are needed in the sequel.

LEMMA A [3,8] *A metric space (X,d) is L_1–embeddable if and only if $(Y, d_{|Y})$ is L_1–embeddable for each finite set of X, where $d_{|Y}$ denotes the restriction of d to the set Y.*

LEMMA B *If (X,d) is an L_1–embeddable metric space then for any points $u, v \in X$ the interval $I(u,v)$ is convex.*

THEOREM C [6] *Let G be a bipartite graph endowed with the shortest–path distance. Then G is h–embeddable if and only if for any adjacent vertices u and v the sets*

$$W(u,v) = \{z \in V : d(u,z) < d(v,z)\}, W(v,u) = \{z \in V : d(v,z) < d(u,z)\}$$

are convex.

This result can be interpretted in the following way. Given two edges $e = (u,v)$ and $e' = (u',v')$ of a bipartite graph G, define $e\theta e'$ if and only if $u' \in W(u,v)$ and $v' \in W(v,u)$. The relation θ is reflexive, symmetric, but not transitive in general. Djokovic [7] proved that G is h–embeddable if and only if θ is transitive.

Let $w = (w_e)_{e \in E}$ be nonnegative weights assigned to the edges of G. The weighting w is said to be *compatible* with the relation θ if $w_e = w_{e'}$ whenever $e\theta e'$.

LEMMA D [4] *Let G be abipartite h–embeddable graph and let w be a weighting of the edges of G which is compatible with the relation θ. Then the resulting metric space is L_1–embeddable.*

Let P be a *rectilinear polygon* in the plane R^2 (i.e. a polygon having all edges axis–parallel). A *rectilinear path* π is a polygonal chain consisting of axis–parallel segments lying inside P. The length of the path π in the rectilinear metric is defined as the sum of the length of the segments π consists of. In other words, the length of π is equal to its Euclidean length. For any two points u and v in P, the *rectilinear distance* between u and v, denoted by $d(u,v)$, is defined as the length of the minimum length rectilinear path connecting u and v [7]. Distance problems on polygons are fundamental in computational geometry and have many applications. A variety of problems such as shortest path queries problems, center and diameter problems, and facility location problems have been studied for various metrics. Rectilinear versions of these problems are motivated by applications in areas, such as VLSI –design, plant and facility layout, urban transportation, wire layout, and robot motion.

Simple rectilinear polygons, being median metric spaces, are L_1–embeddable (for the definition of median spaces and related results consult [9]). Any rectilinear polygon P can be represented as a simple rectilinear polygon inside which lie pairwise disjoint obstacles (holes). Each *hole* represents the interior of a simple rectilinear polygon. In this paper we address the next question:

What rectilinear polygons are L_1-embeddable?

Unfortunately, not all rectilinear polygons with holes are L_1-embeddable. As the next examples show even in the simple cases the L_1-embeddability is not fulfilled.

EXAMPLE 1 Let P be a rectangle with an L-gon as a hole; see Figure 1. Then $x, y \in I(u,v)$, however, $z \in I(x,y) \setminus I(u,v)$. By Lemma B the resulting metric space is not L_1-embeddable.

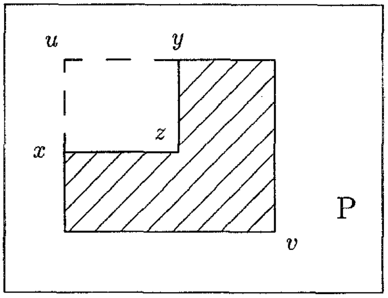

Figure 1

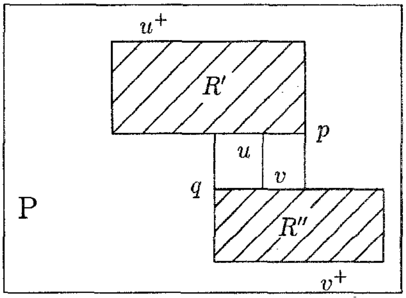

Figure 2

The next example shows that even when all holes are rectangles the L_1-embeddability does not hold.

EXAMPLE 2 Let P be a rectangle with two rectangular holes R' and R'' such that two points $u' \in intR'$ and $v' \in intR''$ have the same x-coordinate; see Figure 2. Assume that the segment $[u'v']$ intersects the rectangles R' and R'' in the points u and v, respectively. We can suppose that $d(u,p) = d(v,q)$, otherwise we can replace u and v by points which fulfill this condition. Denote by u^+ and v^+ the points of R' and R'' opposite to u and v, respectively. Then $p, q \in I(u^+, v^+)$, however, $u, v \in I(p,q) \setminus I(u^+, v^+)$. Again, applying Lemma B we conclude that P is not L_1-embeddable.

Below we present a class of L_1-embeddable rectilinear polygons. An axis-parallel segment is called a *cut segment* of a polygon P if it connects two edges of P and lies entirely inside P. A cut is called a *maximal cut* if it does bot belongs to another cut of P as a proper cut.

THEOREM *Let P be a rectilinear polygon whose all holes are rectangles. If there are no maximal cuts whose end-points lie on boundaries of different holes of P then P is L_1-embeddable.*

PROOF By Lemma A it is sufficient to show that any finite set of points S of P is L_1-embeddable. For S define a grid Γ using the following rules:

(a) take all horizontal and vertical cuts of P which pass through points of S or vertices of P;

(b) for any cut $c = [p, p']$ if p belongs to the hole H then take the point p^+ opposite to p in H and take the cut with the same direction as c and which has p^+ as an end–point.

The cuts defined on steps (a) and (b) together with their intersections points generate the grid Γ. Let $G = (V, E)$ be the graph associated with this grid. (These constructions are illustrated in the Figure 3). Denote by $d_G(u, v)$ the shortest path distance between vertices u and v in the graph G. Let also $I_G(u, v)$ denotes the interval of G between u and v.

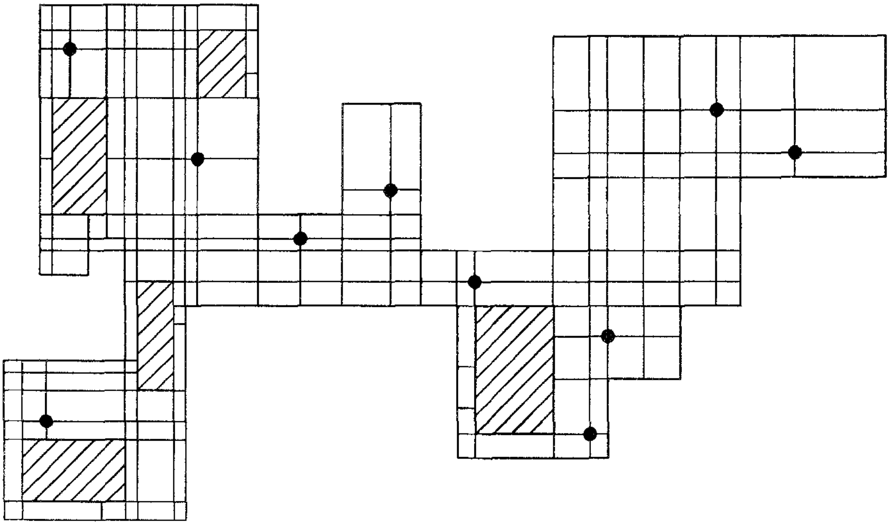

Figure 3

Claim 1 *$d(u, v) = d_{|\Gamma}(u, v)$ for any vertices $u, v \in V$ of the grid Γ.*

Proof Assume the contrary and among the pairs of grid–vertices which violate this condition we choose the vertices u and v at minimum distance (Such choice is possible since there exists a finite number of grid–vertices). Let $Q = (w_0 = u, w_1, \dots, v = w_k)$ be a shortest path between u and v. By our choice we conclude that no points of this path, except u and v, are grid–vertices. Consider the first link $[w_i w_{i+1}]$ of Q which orthogonally intersects some cut c of the grid Γ. Suppose that c is a vertical cut and let $[w_i w_{i+1}] \cap c = \{w\}$. By W^+ we denote the intersection of c with the horizontal cut which passess through u. Then replacing in Q the subpath between u and w by the two–links path (u, w^+, w) we obtain again a shortest path between u and v. This path contains a grid–vertex w^+ and thus $w^+ \in I(u, v)$. By our assumption there exists a shortest path Q' connecting w^+ and v and whose vertices belong to V. Since $w^+ \in I(u, v)$ we conclude that Q' can be extended to a similar shortest path between u and v, i.e. $d(u, v) = d_{|\Gamma}(u, v)$. □

Next we consider some properties of the graph G of the grid Γ. Note that G is a planar graph, all faces of which are 4–cycles, except the faces generated by holes. By the definition of Γ (see rule (b)) we have that every such face is an even cycle. Therefore we obtain the following property of the graph G.

Claim 2 *G is a bipartite graph.*

Given two edges e and e' of the graph G define $e\theta^* e'$ if there exist edges $e_0 = e, e_1, \ldots, e_{k-1}, e_k = e'$ such that e_{i-1} and e_i are opposite edges of a common interior face of G for all $i = 1, \ldots, k$. Evidently, θ^* is an equivalence relation on the edges of G. Let $E_1, \ldots, E_m$ be the classes of equivalent edges of the graph G. It is not difficult to remark that for any E_i the end-vertices of all edges from E_i lie on at most two pairs of parallel cuts of P; see Figure 4.

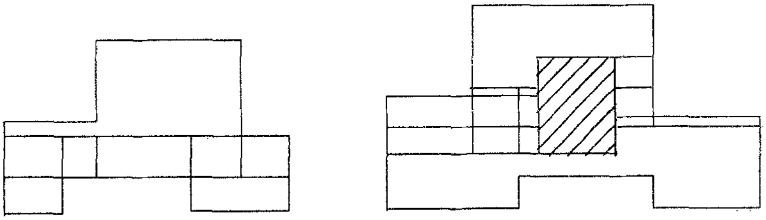

Figure 4

Indeed, if we consider all interior faces of G which contain the edges of E_i then all they are rectangles, except at most one which corresponds to a hole of P. In particular, we obtain

Claim 3 *If $e\theta^* e'$ then the edges e and e' of G does not belong to a common cut of P.*

Claim 4 *If vertices $u, v \in V$ belong to a common cut of P then the interval $I_G(u, v)$ consists of all grid-vertices of the segment $[uv]$.*

Proof Assume the contrary and let T be a shortest path in G whose vertices, except u and v, does not belong to the segment $[uv]$. Let also Q be the path between u and v consisting of grid-vertices from $[uv]$. Any edge e of the path Q is equivalent to some edge of T. Indeed, if say $e \in E_i$, then the cuts which generate the class E_i necessarily intersect the path T. By Claim 3 we obtain that the path T contains at least $|Q|$ parallel edges. Since T have at least two edges perpendicular to $[uv]$ we get $|T| > |Q|$, a contradiction. □

Claim 5 *If u and v are grid-vertices which belong to perpendicular cuts c and c', respectively and $c \cap c' = \{w\}$ then $w \in I_G(u, v)$.*

Proof Let Q and Q' be the shortest paths between u, w and v, w, respectively. Assume that c is horizontal, while c' is vertical. Consider an arbitrary shortest path T between u and v. Using the arguments of Claim 4 we deduce that T contains at least $|Q|$ horizontal edges and at least $|Q'|$ vertical edges. Therefore $w \in I_G(u, v)$. □

Deleting the edges of an equivalence class E_i we obtain two connected components of the graph G. Let V_i^1 and V_i^2 be the vertex-sets of these components.

Claim 6 *For any equivalence class E_i of G both sets V_i^1 and V_i^2 are convex.*

Proof Assume the contrary and among the pairs of vertices which violate the convexity of the set V_i^1 or of the set V_i^2 select the vertices u, v at minimum distance. Let, for example, $u, v \in V_i^1$. Since $I(u,v) \not\subseteq V_i^1$, from the choice of u and v we obtain that there exists a shortest path Q between u and v whose all non-end vertices lie in V_i^2. Denote by u' and v' the neighbours of u and v in this path. Then necessarily $(u,u'), (v,v') \in E_i$. By Claim 4 the vertices u and v does not belong to a common cut of P, i.e. they are "separated" by some hole H; see Figure 5.

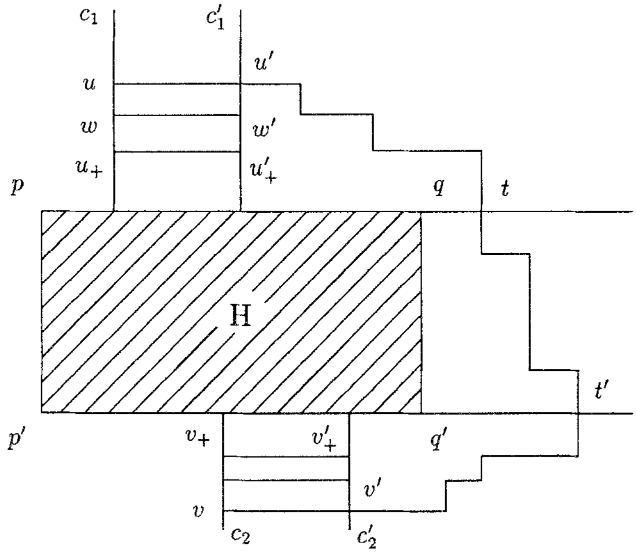

Figure 5

Let p, q, p' and q' be the corners of H. Assume without loss of generality that the vertical cuts which pass through the vertices u, u', v and v' intersect the horizontal sides $[pq]$ and $[p'q']$ of the rectangle H. Denote their intersections by u_+, u_+', v_+ and v_+', respectively; see Figure 5. The horizontal cuts c_h and c_h' which respectively pass through the vertices q and q' intersect the path Q in some vertices t and t'. Evidently, $t, t' \in V$. By Claim 5 $u_+' \in I_G(u', t)$ and $v_+' \in I_G(v', t')$. Therefore we obtain a new shortest path $Q' = (u, u', \ldots, u_+', q, q', v_+', \ldots, v', v)$ between u and v whose vertices does not belong to V_i^1. First assume that $u = u_+$ and $v = v_+$, i.e. both vertices u and v belong to the rectangle H. In this case the paths between u_+', q and v_+, p'; u_+, p and v_+', q'; p, p' and q, q' have the same lengths in G. This, however, contradicts to our assumption that $u', v' \in I(u,v)$. So, assume, for example, that $u \neq u_+$. Let w be the neighbour of u in the path $I(u, u_+)$, while w' denote the neighbour of u' in the path $I(u', u_+')$. Vertices w and w' are adjacent, and moreover $(w, w') \in E_i$. Since $w' \in Q'$ and $w \in I(u, w')$ we get $w \in I(u,v)$. In particular, $d(w,v) < d(u,v)$. Since $(w, w', \ldots, u_+', q, q', v_+', \ldots, v', v)$ is a shortest path in G between w and v and $w' \in V_i^2$ we obtain a contradiction with the choice of vertices u and v. The obtained contradiction show that both sets V_i^1 and V_i^2 are convex for all equivalence classes of G. □

Claim 7 $\theta = \theta^*$.

Proof Let $(u,v) \in E_i$ and (u',v') be an edge of G such that $u' \in W(u,v), v' \in W(v,u)$, however, $(u',v') \notin E_i$. Assume, for example, that $u,u',v' \in V_i^1$, while $v \in V_i^2$. Then $v \in I_G(u,v')$ in contradiction with the convexity of the set V_i^1 (Claim 6).
Conversly, let $(u,v),(u',v') \in E_i$, however, $u',v' \in W(u,v)$. Since G is bipartite then either $u' \in I(u,v')$ or $v' \in I(u,u')$. Let, say $u' \in I(u,v')$. Because $u \in I(u',v) \cap I(v',v)$ we conclude that all vertices v,u,u' and v' lie on a common shortest path between v and v'. Again we obtain a contradiction with Claim 6. □

From this result we obtain that the Djokovic relation θ is transitive. By Theorem C G is h-embeddable. By Lemma D the grid Γ is L_1-embeddable. From Claim 1 we conclude that any finite subset S of P is L_1-embeddable. Finally, by Lemma A the whole polygon P is L_1-embeddable. This concludes the proof of the theorem. □

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