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Peakless functions on graphs

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Abstract

Let $G = (V, E)$ be a finite connected graph, endowed with the standard graph-metric $d(u, v)$. A real valued function f defined on V is called *peakless* if $d(u, w) + d(w, v) = d(u, v)$ implies $f(w) \leqslant \max\{f(u), f(v)\}$ and equality holds only if $f(u) = f(v)$ (see Busemann, (1955) for the general definition). Peakless functions inherit and generalize the properties of usual convex functions.

In this paper we investigate the properties of peakless functions in graphs. We define a convexity in graphs, known in the geometry of geodesics under the name "total convexity", which is closely related with peakless functions. Namely, totally convex sets are precisely the level sets of peakless functions. In particular, a graph has no nonconstant peakless functions if and only if it does not contain proper totally convex sets. We call such graphs peakless-prime and show that an arbitrary graph G admits a decomposition in which all members are peakless-prime. These primes are exactly the subgraphs of G on which all peakless functions of G are constant. It is interesting to notice that such a decomposition into peakless-prime subgraphs represents a common modular and simplicial decomposition of G.

Keywords: Graphs; Geodesics; Peakless functions; Maximum Cardinality Search.

1. Introduction

Let (X, ρ) be a metric space. According to [2, p.109], a real valued function f defined on X is called *peakless* if $\rho(u, w) + \rho(w, v) = \rho(u, v)$ implies $f(w) \leqslant \max\{f(u), f(v)\}$ and equality holds only if $f(u) = f(v)$. Properties of peakless functions in G-spaces and G-surfaces were previously studied by Busemann [2], Busemann and Phadke [3], and Inammi [14]. Peakless functions inherit and generalize the properties of usual convex functions (one of them is their unimodality). Among the peakless functions one finds the convex functions in metric spaces, defined and studied by Dearing et al. [5], Soltan and Soltan [23], and Soltan [21]. Convex functions in graphs and other classes of functions related to convexity have been considered by Soltan and Chepoi [24] and Chepoi [4].

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Now, the purpose of the presented paper is to investigate the properties of peakless functions in graphs. We define a convexity in graphs, known in the geometry of geodesics under the name “total convexity”, which is closely related with peakless functions. Namely, totally convex sets are precisely the level sets of peakless functions. In particular, a graph has no nonconstant peakless functions if and only if it does not contain proper totally convex sets. We call such graphs peakless-prime and show that an arbitrary graph G admits a decomposition in which all members are peakless-prime. These primes are exactly the subgraphs of G on which all peakless functions of G are constant. It is interesting to notice that such a decomposition into peakless–prime subgraphs represents a common modular and simplicial decomposition of G; each of these concepts have been already separately investigated in [6–9, 11, 12, 15, 17–19]. The present paper also improve the results of the unpublished manuscript [4].

2. Basic concepts

We begin with some terminology. In what follows let $G=(V,E)$ denote a finite connected simple (i.e. without loops and multiple edges) and undirected graph. For $M \subseteq V$ let $G(M)$ be the *subgraph induced* by M. The *distance* $d(u,v)$ between vertices $u,v \in V$ is the length (i.e. number of edges) of a *shortest path* connecting u and v.

A *walk* is a sequence of vertices $W=(w_0,w_1,\dots,w_p)$ such that w_{i-1} and w_i are adjacent for all $i \in \{1,\dots,p\}$. Then p is called the *length* of W. Following the adopted geometric terminology, by a *geodesic* we will mean a locally–shortest walk, i.e. a walk $(w_0,w_1,\dots,w_p)$ in which $d(w_{i-1},w_{i+1})=2$ for all $i \in \{1,\dots,p-1\}$. By a *geodesic loop* is meant a geodesic whose end-vertices coincide. Any induced path or a shortest path is a geodesic. The converse is not always true. By general agreement, a set $M \subseteq V$ is called *totally convex* [2, 3] if it contains every geodesic whose end-vertices are in M. For $X \subset V$ the intersection H of all totally convex subsets of G containing X is again totally convex in G; H will be called the *convex hull* of X in G and be denoted by *conv*(X). Recall also that a set $M \subseteq V$ is *convex* if M contains every shortest path in G whose end-vertices are in M.

Let $W=(w_0,w_1,\dots,w_p)$ be a walk of a graph G. A real valued function f defined on W is *peakless* if $0 \leqslant j < i < k \leqslant p$ implies $f(w_i) \leqslant \max\{f(w_j),f(w_k)\}$ and equality holds only if $f(w_j)=f(w_k)$.

Next we consider only real valued functions f defined on vertices of a graph G. For $M \subset V$ by $f|_M$ we denote the *restriction* of f on M. For a real number α let

$$[f \leqslant \alpha]=\{v \in V:\ f(v) \leqslant \alpha\}, \qquad [f=\alpha]=\{v \in V:\ f(v)=\alpha\}$$

be the *level sets* of f. In a similar way we can define the sets $[f<\alpha]$, $[f \geqslant \alpha]$ and $[f>\alpha]$.

A function f defined on G is *peakless* [2, 3] if the restriction of f on any shortest path of G is peakless. In the sequel, by $\mathscr{P}(G)$ we denote the family of all peakless functions of G. The following evident lemmas capture some basic properties of peakless functions.

Lemma 1. *For a function f defined on a graph G the following conditions are equivalent:*

(i) *f is peakless;*

(ii) *f is locally-peakless, i.e. f is peakless on any induced path of length two;*

(iii) *the restriction of f on any geodesic of G is peakless.*

Proof. Implications (i) $\rightarrow$ (ii) and (iii) $\rightarrow$ (i) are trivial. In order to show that (ii) $\rightarrow$ (iii) consider an arbitrary geodesic $(w_0, w_1, \ldots, w_p)$ and suppose that $f(w_i) = \max\{f(w_j)\colon\ 0 < j < p\}$. Assume that $f(w_i) \geqslant \max\{f(w_0), f(w_p)\}$, otherwise we are done. Since $f(w_i) \geqslant \max\{f(w_{i-1}), f(w_{i+1})\}$ from the choice of w_i, and since f is locally-peakless, necessarily $f(w_{i-1}) = f(w_i) = f(w_{i+1})$. Then, by letting w_{i-1} play the role of w_i and w_{i-2} and w_i the roles of w_{i-1} and w_{i+1}, respectively, we obtain the equality $f(w_{i-2}) = f(w_{i-1}) = f(w_i)$. And so on, until we arrive at the vertex w_1. Hence $f(w_0) = f(w_1) = \cdots f(w_{i-1}) = f(w_i)$. Consequently, applying the same procedure to the vertices $w_{i+1}, w_{i+2}, \ldots, w_{p-1}, w_p$ we get $f(w_i) = f(w_{i+1}) \ldots = f(w_{p-1}) = f(w_p)$. □

Lemma 2. *If f is a peakless function of G then each level set $[f \leqslant \alpha]$ is totally convex. In particular, $[f \leqslant \alpha]$ is convex.*

Lemma 3. *If f is peakless on G and H is a connected induced subgraph of G then the function $f|_H$ is peakless on H.*

3. Peakless-prime graphs

A graph G will be called *peakless-prime* (with respect to the family of peakless functions) if it has no nonconstant peakless functions. In this section we characterize peakless-prime graphs. Recall that a *module* (or a *homogeneous set*) M in a graph G is a proper subset of V such that every vertex of $V \backslash M$ is adjacent to either all or none of the vertices of M [18, 19]. By a *simplicial module* is meant a module M of G whose neighbourhood in $V \backslash M$ induces a simplex (i.e. a complete subgraph). Recall that the *neighbourhood* of M in $V \backslash M$ consists of all vertices of $V \backslash M$ adjacent to vertices of M.

Define on the set $\mathscr{P}(G)$ of peakless functions of a graph G a partial order $\prec$: if $f, g \in \mathscr{P}(G)$ then $f \prec g$ if and only if each level set $[f \leqslant \alpha]$ of f is a level set of g. Two functions $f, g \in \mathscr{P}(G)$ are called *equivalent* if simultaneously $f \prec g$ and $g \prec f$ hold, i.e. they have the same level sets.

Let G be a nonprime graph and let f be a peakless function on G whose maximum value is α^*. Then the set $[f < \alpha^*]$ is called a *maximum level set* of G. Let us note a few straightforward implications of these definitions.

Lemma 4. *Let L be a maximum level set of a graph G and let ψ be the characteristic function of the set $V \setminus L$, i.e. $\psi(v) = 0$ if $v \in L$ and $\psi(v) = 1$ otherwise. Then ψ is peakless.*

Lemma 5. *Let f be a peakless function on G such that the subgraph $G([f = \alpha])$ is not a peakless-prime graph for some real number α. Then there exists a nonconstant on $[f = \alpha]$ peakless function $g \in \mathscr{P}(G)$ such that $f \prec g$.*

Proof. For convenience let $\alpha = \alpha_i$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$ are the values of f. Let L be a maximum level set of the graph $G([f = \alpha_i])$. Define the function g using the following rules: if $v \in [f < \alpha_i] \cup L$, then put $g(v) = f(v)$, otherwise, if $v \notin L$ and $v \in [f = \alpha_{i+j}]$, $j \in \{0, \dots, n-i\}$, then put $g(v) = f(v) + j + 1$. The resulting function g has the same level sets as f, except the set $[g \leqslant \alpha_i] = [g \leqslant \alpha_{i-1}] \cup L$. Hence $f \prec g$. A direct application of Lemmas 1 and 4 implies that g is peakless. □

Lemma 6. *A function f is peakless on G if and only if for any real number α the set $[f = \alpha]$ is either empty or a simplicial module in the subgraph $G([f \leqslant \alpha])$.*

Theorem 1. *For a graph G the following conditions are equivalent:*
(i) *G is a peakless-prime graph;*
(ii) *G has no simplicial modules;*
(iii) *G has no proper totally convex sets.*

Proof. (i) → (ii) follows from Lemma 6, while (iii) → (i) is a consequence of Lemma 2.

(ii) → (iii): Suppose the contrary: then there is a proper maximal by inclusion totally convex set M of G. We claim that $V \setminus M$ is a simplicial module of G. Let y be a vertex of $V \setminus M$ adjacent to some vertex $x \in M$. In order to prove our assertion we have to show that x is adjacent to any other vertex $z \in V \setminus M$. By maximality of M we have $conv(M \cup y) = V$, in particular $z \in conv(M \cup y)$.

The total convexity is a particular case of interval convexities [22, 26], where the *interval* $I(u,v)$ between two vertices u and v is the union of all geodesics having u and v as end-vertices. Recall that the convex hull of any set $A \subset V$ may be constructed in the following way (consult [22, 26]):

$$conv(A) = \bigcup_{i \geqslant 0} P^i(A), \quad \text{where } P(A) = \bigcup \{I(u,v)\colon\ u, v \in A\},$$

$$P^0(A) = A, \qquad P^{i+1}(A) = P(P^i(A)), \quad i = 1, 2, \dots.$$

As $z \in conv(M \cup y)$, then for some integer $k \geqslant 1$ we have $z \in P^k(M \cup y)$. We prove the adjacency of vertices x and z by induction on k. First assume that $k = 1$. Then z belongs either to a geodesic $L(y,v)$ connecting y and a vertex $v \in M$ or to a geodesic loop $L(y,y)$ with the end-vertices in y.

First suppose that $z \in L(y,v)$. Without loss of generality we can assume that $L(y,v) \cap M = \{v\}$. The vertex x must be adjacent to the neighbour y' of y in $L(y,v)$, otherwise the walk $(x,y,y',\ldots,z)$ is a geodesic, contrary to total convexity of M. Continuing this way we conclude that x will be adjacent to all vertices of $L(y,v)$, in particular x and z are adjacent.

Next consider the case when $z \in L(y,y) = (y_0 = y, y_1, \ldots, y_m, y_0 = y)$, where $z = y_i$. Again we may assume that $L(y,y)$ does not intersect the set M, otherwise we become in the conditions of the first case. Among the vertices of $L(y,y)$ nonadjacent to x let y_t has the smallest index. Necessarily $t < i$, otherwise x and z are adjacent and we are done. Let y_l be the first vertex of $L(y,y)$ after $z = y_i$ which is adjacent to x. Then we obtain a geodesic $(x, y_{t-1}, y_t, \ldots, y_i, \ldots, y_l, x)$, whose all vertices except x do not belong to the set M. Again we obtain a contradiction with total convexity of M.

Next, in the assumption that all vertices of the set $P^k(M \cup y)\backslash M$ are adjacent to x, pick an arbitrary vertex $z \in P^{k+1}(M \cup y)\backslash P^k(M \cup y)$. Then z belongs to a geodesic $L(z_1,z_2)$ between two vertices $z_1, z_2 \in P^k(M \cup y)$, where $z_2 \notin P^{k-1}(M \cup y)$. Then, by virtue of the induction hypothesis, the vertices x and z_2 are adjacent. We distinguish two cases: either $z_1 \in M$ or $z_1 \notin M$. In the either case x must be adjacent to all vertices of $L(z_1,z_2)$, otherwise M is not totally convex. Otherwise, if $z_1 \notin M$, then x is adjacent to both z_1 and z_2. If x and z were not adjacent, then we obtain a geodesic $L(x,x) \subseteq L(z_1,z_2) \cup \{x\}$ whose all vertices except x are outside M, contrary to the total convexity of M. Hence, $V\backslash M$ is a module of G. Since M is totally convex, $V\backslash M$ is a simplicial module. □

4. Level sets

In this section we study the structure of level sets of peakless functions. We say that two vertices x and y of a graph G are *equivalent* if for any peakless function f we have $f(x) = f(y)$. Then G can be represented as a disjoint union of sets of equivalent vertices. By *cells* we will mean the subgraphs induced by these sets. Let $C_1, \ldots, C_n$ be the cells of the graph G.

Lemma 7. *Every cell of a graph G is a peakless-prime graph.*

Proof. Consider a peakless function f which takes the maximum number of different values $\alpha_1 < \cdots < \alpha_n$. We assert that each set $[f = \alpha_i]$ induces a cell. Suppose this fails and let $[f = \alpha_i] = C_{i_1} \cup \cdots \cup C_{i_k}$. By Lemma 3 the restriction on $G([f = \alpha_i])$ of any peakless function is peakless on each of its connected components. Consequently, we conclude that the graph $G([f = \alpha_i])$ is not peakless-prime. By Lemma 5 there exists a peakless function $g \in \mathscr{P}(G)$ such that $f \prec g$. This, however, is absurd by the choice of f. So, $G([f = \alpha_i])$ is a cell for each value α_i. If some of the cells were not peakless-prime, then the repeated application of Lemma 5 leads us to a contradiction with the choice of f. □

From Theorem 1 and Lemma 1 we obtain the following

Corollary 1. *If a totally convex set M intersects the cell C_i then $C_i \subseteq M$.*

Let f be a function defined on a subset M of a graph G. We say that a function g is an *extension* of f if $g|_M = f$.

A function f defined on G *separates* the disjoint sets $S_1, \ldots, S_m$ if for any $1 \leqslant i < j \leqslant m$ one of the inequalities

$$\min\{f(v):\ v \in S_i\} > \max\{f(v):\ v \in S_j\}, \qquad (1)$$

$$\min\{f(v):\ v \in S_j\} > \max\{f(v):\ v \in S_i\} \qquad (2)$$

holds.

Theorem 2. *Let f be a peakless function defined on a totally convex set M of a graph G. Then there exists a peakless extension f^* of f to the whole graph G, such that M is a level set of f^* and f^* separates the sets $M, C_{i_1}, \ldots, C_{i_k}$, where $C_{i_1}, \ldots, C_{i_k}$ are the cells disjoint from M.*

Proof. Proceed by induction on the number n of cells of the graph G. For $n = 1$ there is nothing to show, because G is a peakless-prime graph, and by virtue of Theorem 1 G does not contain proper totally convex sets. Assume that the assertion is true for all graphs with at most $n - 1$ cells, and consider an arbitrary graph G with n cells $C_1, \ldots, C_n$. From the proof of Lemma 7 we know that the peakless function on G which has the maximum number of different values separates the cells of G. Suppose without loss of generality that C_n is the cell on which this function attains the maximum value. Then C_n is a simplicial module of G. According to Corollary 1 we distinguish two cases: $C_n \subseteq M$ or $C_n \cap M = \emptyset$.

First suppose that $C_n \cap M = \emptyset$. The graph $G(V \backslash C_n)$ is connected and M is totally convex in this graph. By the induction assumption there exists an extension f^* of f to the graph $G(V \backslash C_n)$ which satisfies the theorem's conditions. Extend the function f^* to the whole graph G by putting $f^*(v) = \max\{f^*(x):\ x \in V \backslash C_n\} + 1$ for any vertex $v \in C_n$. Since C_n is a simplicial module, f^* is a desired peakless function on G.

Next consider the case when $C_n \subseteq M$. Let $N(C_n)$ be the neighbourhood of C_n in the set $V \backslash C_n$, i.e.

$$N(C_n) = \{v \notin C_n:\ \exists u \in C_n\ (u, v) \in E\}.$$

We claim that the set $M^* = (M \backslash C_n) \cup N(C_n)$ is totally convex in both graphs G and $G(V \backslash C_n)$. Suppose the contrary: then there exists a geodesic $L(v_1, v_2)$ with the end-vertices in M^* and whose all interior vertices are outside $M^* \cup C_n$. Since $M \backslash C_n$ is totally convex, necessarily at least one of the vertices v_1 or v_2, say v_2, must be in $N(C_n) \backslash M$. If $v_1 \in N(C_n) \backslash M$ then for arbitrary vertex $w \in C_n$ the walk $(w, L(v_1, v_2), w)$ is a geodesic loop. Otherwise, if $v_1 \in M$, then $(w, L(v_2, v_1))$ is a geodesic. In both cases we obtain a contradiction with total convexity of the set M. Thus M^* is totally convex.

For a vertex $v \in N(C_n)\backslash M$ let $C(v)$ be the cell of G which contains v. We assert that $C(v) = \{v\}$. Otherwise, if $C(v) \neq \{v\}$, then by Theorem 1 there is a geodesic loop $L(v,v) \subseteq C(v)$ with the end-vertices in v. Let w be a vertex from C_n adjacent to v. Since $v \notin M$, by Corollary 1 we deduce that $M \cap C_n = \emptyset$. Necessarily w is adjacent to all vertices of $L(v,v)$, for otherwise we get a geodesic loop $L(w,w) \subseteq \{w\} \cup L(v,v)$ with the end-vertices in w, contradicting to total convexity of M. But then $L(v,v) \subset N(C_n)$, contrary to the initial choice of C_n as a simplicial module. Thus $C(v) = \{v\}$ for any vertex $v \in N(C_n)\backslash M$.

The set $M\backslash C_n$ is totally convex in the graph $G(M^*)$. Applying the induction hypothesis to this graph and to the function f, we get a peakless function f' on $G(M^*)$ which satisfies the theorem's conditions. If $M = C_n$, then f' can be any function separating the vertices of the set $N(C_n)$. We can further extend the function f' to a peaklesss function f'' on $G(V\backslash C_n)$ which satisfies all conditions of the theorem, too. Assume that f'' takes the values $\alpha_1 < \cdots < \alpha_p < \cdots < \alpha_t < \cdots < \alpha_m$, where

$$M\backslash C_n = \{v \in V\backslash C_n\colon\ f''(v) \leqslant \alpha_p\}, \qquad M^* = \{v \in V\backslash C_n\colon\ f''(v) \leqslant \alpha_t\}.$$

Suppose that on the cell C_n the function f takes the value α. Now we wish to extend the function f'' to the whole graph G. If $\alpha \leqslant \alpha_p$ then we simply put $f^*(v) = f''(v)$ if $v \notin C_n$, and $f^*(v) = \alpha$ if $v \in C_n$. Otherwise, if $\alpha > \alpha_p$, then put $f^*(v) = f''(v) + \alpha - \alpha_p$ if $v \in M$, and $f^*(v) = f(v)$ for any $v \notin M$. Clearly M is a level set of f^*, because $M = [f^* \leqslant \max\{\alpha_p, \alpha\}]$. Since f'' separates the sets $M\backslash C_n, C_{i_1}, \ldots, C_{i_k}$, necessarily f^* separates the sets $M, C_{i_1}, \ldots, C_{i_k}$. So, we have to show only that f^* is peakless. According to Lemma 1 it suffices to establish that f^* is locally-peakless. Let v_1 and v_2 be two vertices at distance two and let v be their common neigbour. Since f^* and f coincide on M and f^* and f'' are equivalent on the set $V\backslash C_n$, we can consider only the case when $v_1 \in C_n$ and $v_2 \notin M$. Then evidently $v \in N(C_n)\backslash M$. As we already proved $C(v) = \{v\}$, and therefore $v_2 \notin C(v)$. Moreover, since C_n is a simplicial module and v_1 is adjacent to all vertices of $N(C_n)$, we conclude that $v_2 \notin N(C_n)$. Recall that M^* is a level set of the function f''. This means that $f^*(v) = f''(v) < f''(v_2) = f^*(v_2)$. Therefore $f^*(v) < \max\{f^*(v_1), f^*(v_2)\}$, thus completing the proof of Theorem 2. □

Corollary 2. *A set $M \subset V$ of a graph G is a level set $[f \leqslant \alpha]$ of some peakless function f if and only if M is totally convex.*

Corollary 3. *Any peakless function defined on a totally convex set of G can be extended to a peakless function on the whole graph G.*

A graph G is called *chordal* [1] if G has no induced cycles of length greater than three. In chordal graphs all geodesics are induced paths, while totally convex sets are exactly the induced-path convex sets. (Recall, that a set is called *induced-path convex* if together with any two vertices it contains all induced paths connecting these vertices.) Since any peakless function is constant on any induced cycle of length at most four, in chordal graphs all cells are one-vertex subgraphs. This leads us to the following result.

Corollary 4. (i) *There exists a peakless function which separates the cells of an arbitrary graph G.*

(ii) *There exists a peakless function which separates the vertices of a graph G if and only if G is chordal.*

5. Extremal structure of totally convex sets

In this section we establish further connections between peakless functions and totally convex sets. We interpret the cells as the minimal faces of totally convex sets and show that any totally convex set is the convex hull of its extremal cells. This means that the total convexity is a convex geometry [16]. This result thus entails similar theorems for induced-path convexity [10], and convexity [10, 16, 22].

Following the terminology of abstract convexity [22, 26], a subset F of a set M is called a *face* of M provided it satisfies the following property: if $z \in F$ and $z \in I(x, y)$ for some vertices $x, y \in M$ distinct from z then $x, y \in F$. Recall that $I(x, y)$ denotes the interval with x and y as end-vertices (for total convexity the interval $I(x, y)$ is the union of all geodesics between x and y). One-vertex faces are called *extremal vertices.* It is easy to show that respect to all three types of convexity (convexity, induced-path convexity and total convexity) a vertex is extremal in M if and only if it is *simplicial,* i.e. its neighbours in M induce a complete subgraph. The following properties of faces are evident; see [22]:

(i) The union and the intersection of faces of a set M are faces too

(ii) If F is a face of S and S is a face of M then F is a face of M

(iii) If $F \subset S \subset M$ and F is a face of M then F is a face of S

(iv) Let $F_1 \subset F_2 \subset \cdots \subset F_n$ be a chain of sets such that for each pair $i < j$ F_i is a face of F_j. Then $\bigcap_{i=1}^{n} F_i$ is a face of any set F_i.

The next result is a reformulation of a similar result for convex functions [22] and usual convex sets.

Lemma 8. *Let f be a peakless function on a graph G. Then for any subset $M \subset V$ the set $\{v \in M:\ f(v) = \max f(u), u \in M\}$ is a face of M with respect to the total convexity.*

A face F of a set M is called *minimal* if F does not contain proper faces. By properties (i)–(iv) we conclude that the face F is minimal if and only if F is a minimal by inclusion face of M.

Lemma 9. *Let M be a totally convex set of a graph G. Then any minimal face of M is a cell of G.*

Proof. Let F be a minimal face of M. First notice that if F intersects a cell C_i, then necessarily $C_i \subset F$. Indeed, if we suppose the contrary, then the set $C_i \backslash F$ is totally convex, in contradiction with Theorem 1 and Lemma 7. Consider a peakless function f separating the cells of G. Let $F' = \{v \in F\colon\ f(v) = \max f(u),\ u \in F\}$. The subgraph induced by F' is a cell of G. On the other hand, from Lemma 8 we know that F' is a face of both sets F and M. Thus $F' = F$, i.e. F is a cell of G. □

Let M be a totally convex set of G. Denote by $Ext(M)$ the set of all *extremal cells* of M, i.e. cells induced by minimal faces of M. Let also $ext(M)$ denote the set of all extremal vertices of M.

Recall that a graph G is called *Ptolemaic* [13] if for any four vertices $u, v, w, x \in V$ the following inequality holds

$$d(u,v)d(w,x) \leqslant d(u,x)d(v,w) + d(u,w)d(x,v).$$

It is well known [13] that G is Ptolemaic if and only if G is chordal and any induced path of G is a shortest path. Therefore, in Ptolemaic graphs all three types of convexity coincide.

Theorem 3. (i) *Any totally convex set M of a graph G is the convex hull of its extremal cells. Moreover, if $Ext(M) = \{C_{i_1}, \ldots, C_{i_p}\}$ then $M = conv(v_1, \ldots, v_p)$ for arbitrary vertices $v_1 \in C_{i_1}, \ldots, v_p \in C_{i_p}$.*

(ii) *Any induced-path convex set of G is the convex hull of its extremal vertices if and only if G is chordal* [10].

(iii) *Any convex set of G is the convex hull of its extremal vertices if and only if G is Ptolemaic* [10, 22].

Proof. (i) Let f be a peakless function which separates the cells of G. By Lemmas 8 and 9 the set $\{v \in M\colon\ f(v) = \max\{f(u),\ u \in M\}\}$ induces an extremal cell of M, thus $Ext(M) \neq \emptyset$. Let $M^* = conv(Ext(M))$ and suppose that $M \neq M^*$. Put $M \backslash M^* = C_{i_1} \cup \cdots \cup C_{i_p}$; see Corollary 1. By Theorem 2 there is a peakless function f^* which separates the sets $M^*, C_{i_1}, \ldots, C_{i_p}$ and, in addition, M^* is a level set of f^*. Then, applying again Lemmas 8 and 9, we get the extremal cells which belong to $M \backslash M^*$, a contradiction. This establishes the first part of (i). Any cell $C_{i_j} \in Ext(M)$ has no proper totally convex sets, thus $C_{i_j} = conv(v_{i_j})$ for any vertex v_{i_j} of C_{i_j}.

(ii) and (iii): As we already mentioned, in a chordal graph G all cells are vertices, thus $Ext(M) = ext(M)$ for any subset M of G. Since total convexity and induced-path convexity coincide, we obtain (ii). If, in addition, G is Ptolemaic, then convexity and induced-path convexity coincide, thus we get (iii). □

6. Simplicial-modular decompositions

Decompositions of graphs into "primes" have successfully been applied in various branches of graph theory and elsewhere. Two of them, namely, the modular decomposition and the simplicial decomposition, are closely related to the subject of our paper.

The modular decomposition has been discovered independently by researchers in many areas (see [18] and [19]). The *modular decomposition* is a partition $\mathscr{A}=(A_i)$, $0 < i \leqslant n$, of a graph G into subgraphs, such that each A_i is a module. A graph is called prime with respect to the modular decomposition if it has no nontrivial module [17]. For further informations and applications of modular decompositions consult [15, 17–19]. The family $\mathscr{B}^* = (B_i^*)$, $0 < i \leqslant n$, of induced subgraphs of a graph G is called a *simplicial decomposition* of G [6–9, 11, 12] if thc following conditions hold

(S1) $G = \cup_{i=1}^{n} B_i^*$

(S2) $\left(\bigcup_{j<i} B_j^*\right) \cap B_i^* = S_i$ is a complete graph for each i $(0 < i \leqslant n)$

(S3) No S_i contains B_i^* or any other $B_j^* (0 < j < i \leqslant n)$.

A graph which has no simplicial decomposition into more than one subgraph is called prime [6, 7, 11, 12]. It is known that any finite graph has a simplicial decomposition into primes, and these primes are essentially its smallest induced-path convex subgraphs [6, 7, 11, 12]; for the case of infinite graphs see [6–9, 11, 12]. In particular, chordal graphs have simplicial decompositions whose primes are simplices (complete subgraphs) [1]; such decompositions are usually called *perfect elimination orderings* [20, 25]. It is easy to transform any simplicial decomposition $B^* = (B_i^*)$, $0 < i \leqslant n$, into a partition $\mathscr{B} = (B_i)$, $0 < i \leqslant n$, of G by letting $B_i = B_i^* \backslash S_i$, $0 < i \leqslant n$. Therefore, both modular and simplicial decompositions further will be considered as partitions of a graph.

Combining these two types of decompositions we obtain a new one, called further a simplicial-modular decomposition. Namely, a family $\mathscr{P}^* = (P_i^*)$, $0 < i \leqslant n$, of induced subgraphs of a graph G is called a *simplicial-modular decomposition* of G if the following conditions hold

(P1) $\mathscr{P}^* = (P_i^*)$, $0 < i \leqslant n$, is a simplicial decomposition of G;

(P2) $\mathscr{P} = (P_i)$, $0 < i \leqslant n$, where $P_i = P_i^* \backslash \cup_{j<i} P_j^*$, is a modular decomposition of G.

By a *level set* of a simplicial-modular decomposition we will mean a set $\bigcup_{j \leqslant i} P_j^*$, where $i \leqslant n$. Equivalently, the partition $\mathscr{P}=(P_i), 0 < i \leqslant n$, of G is a simplicial-modular decomposition of G if each P_i is a simplicial module in the subgraph $G(\bigcup_{j \leqslant i} P_i)$. A graph which has no simplicial-modular decomposition into more than one subgraph is called *prime.* Our next goal is to show that the prime graphs are exactly the peakless-prime graphs. Then, by Theorem 2 and its corollaries we immediately get that each graph G admits a simplicial-modular decomposition into primes. We conclude with a polynomial-time algorithm for computing such a decomposition. We start with a few simple observations.

Lemma 10. *Let $\mathscr{P} = (P_i)$, $0 < i \leqslant n$, be a simplicial-modular decomposition of a graph G. Then the function f, where $f(v) = i$ for each $v \in P_i$, is peakless in G.*

Lemma 11. *Any level set of a simplicial-modular decomposition is a totally convex set. Moreover, any totally convex set M is a level set of some simplicial-modular decomposition of G.*

Proof. The first part follows from Lemmas 10 and 2. In order to prove the second part, consider an arbitrary totally convex set M of G. Put $V \backslash M = C_{i_1} \cup \cdots \cup C_{i_p}$, where $C_{i_1}, \ldots, C_{i_k}$ are cells of G. Let f be a peakless function constant on M and which separates the sets $M, C_{i_1}, \ldots, C_{i_k}$ (see Theorem 2). Assume without loss of generality that

$$M = [f = \alpha_1],\ \ C_{i_1} = [f = \alpha_2], \ldots, C_{i_k} = [f = \alpha_{k+1}],$$

where $\alpha_1 < \alpha_2 < \cdots < \alpha_{k+1}$ are the values of f. Then, letting $P_1 = M, P_2 = C_{i_1}, \ldots, P_{k+1} = C_k$ we obtain a decomposition of G. Consequently, since f is peakless, necessarily this decomposition is simplicial-modular. □

Lemma 12. *A graph G is prime if and only if G is peakless-prime.*

Indeed, if G is prime, then by Lemma 11 G has no proper totally convex sets. By Theorem 1 G is peakless-prime. The converse is immediate.

Theorem 4. *Any graph G admits a simplicial decomposition into primes. The primes are the cells of G.*

Proof. The second part of the theorem follows from Theorem 1 and Lemmas 12 and 6. In order to obtain a simplicial-modular decomposition of G into primes consider a peakless function f which separates the cells of G; see Corollary 4. Let $\alpha_1 < \ldots < \alpha_n$ be the values of f. Consider a decomposition $\mathscr{P} = (P_i), 0 < i \leqslant n$, where $P_i = [f = \alpha_i]$. By Lemma 6 each set P_i is a simplicial module in the subgraph $G(\bigcup_{j \leqslant i}[f = \alpha_j])$. Hence, $\mathscr{P}$ is a simplicial-modular decomposition of G. □

In order to find a simplicial-modular decomposition of an arbitrary graph G into primes we present a special numbering of the vertices of G. The algorithm exploits the idea of the *maximum cardinality search* (MCS) *algorithm* of [25] for finding a perfect elimination ordering of chordal graphs. According to MCS the vertices of a graph are numbered in the following way: as the next vertex to number select a vertex adjacent with the largest number of previously numbered vertices, breaking ties arbitrarily [25]. In our algorithm first we compute the (totally) convex hulls of all vertices, and select a vertex whose convex hull has the smallest size and number all vertices of this set with 1. In the next step we select the vertices adjacent with the largest number of previously numbered vertices. Among these vertices we choose a vertex whose convex hull contains the smallest number of previously numbered vertices, breaking ties arbitrarily. Number all unnumbered vertices of this set with minimal positive integer which is still free. We call this procedure the *maxmin cardinality search* (MMCS). If G is chordal, then we obtain the maximum cardinality search. Subsequently we prove that MMCS gives a prime simplicial-modular decomposition of a graph G in polynomial time. Below we reproduce the details of MMCS.

procedure MMCS (Find a simplicial-modular decomposition of G into primes)
Input: A connected finite graph $G=(V,E)$.
Output: A simplicial-modular decomposition of G into primes.

(0) initially all vertices $v \in V$ are unnumbered;
(1) find the (totally) convex hull of all vertices of G;
(2) choose the vertex $v \in V$ with the smallest convex hull and number all vertices of $conv(v)$ with 1, i.e. $P_1 := conv(v)$ and let $M := P_1$;

repeat

(3) among the unnumbered vertices select the vertices which are adjacent to the largest number of previously numbered vertices;
(4) among the selected vertices choose a vertex v whose convex hull contains the smallest number of unnumbered vertices;
(5) number the vertex v and all unnumbered vertices from $conv(v)$ with the minimal possible positive integer i which is still free;
(6) let $P_i := conv(v) \setminus M$ and $M := M \cup P_i$;

until all vertices are numbered

Theorem 5. *The algorithm MMCS finds in polynomial time a simplicial-modular decomposition of a graph G into primes.*

The proof of this theorem requires some auxiliary results.

Lemma 13. *Let M be a totally convex set and let v be a vertex outside M which has the maximum number of neighbours in M. Then the set $M \cup conv(v)$ is totally convex and the set $P = conv(v) \setminus M$ is a module of $G(M \cup conv(v))$.*

Proof. First we prove that the set P is a module of $G(M \cup conv(v))$, i.e. all vertices of P have one and the same neighbourhood in M. By the choice of the vertex v it suffices to show that any vertex $w \in M$ adjacent to v is adjacent to any vertex z of P. As we already noticed in the proof of Theorem 1, the convex hull $conv(v)$ can be constructed step by step, by adding all geodesics between vertices already included in $conv(v)$. Assume that z was included in $conv(v)$ on step k. To verify that z and w are adjacent, proceed by induction on k. First suppose that $k = 1$, i.e. z belongs to a geodesic loop $L(v,v)$ with the end-vertices in v. Suppose by way of contradiction that vertices z and w are nonadjacent. Let z_1 and z_2 be the closest to z vertices adjacent to w and belonging to two subgeodesics of $L(v,v)$, which connect the vertices v and z.

Then the walk $(w, z_1, \ldots, z, \ldots, z_2, w)$ is a geodesic loop with the end-vertices in w. Since $z \notin M$, we obtain a contradiction with total convexity of M. Thus, the vertices z and w are adjacent.

Next assume that $k > 1$, i.e. the vertex z belongs to a geodesic $L(v_1, v_2)$ whose end-vertices v_1 and v_2 were already included in $conv(v)$. Since M is totally convex and $z \notin M$, necessarily at least one of the vertices v_1 or v_2, say v_1, does not belong to M. By the induction hypothesis, the vertices w and v_1 are adjacent. First suppose that $v_2 \in M$. If z and w are nonadjacent and z_1 is the closest to z neighbour of w in the geodesic $L(z, v_1) \subset L(v_1, v_2)$, then we obtain a geodesic $(w, z_1, \ldots, z, \ldots, v_2)$ between w and v_2, contrary to total convexity of M. Next consider the case when $v_2 \notin M$. The vertices w and v_2 are adjacent by virtue of the induction assumption. As in the case $k = 1$, by z_1 and z_2 we denote the closest to z neighbours of w in two subgeodesics of $L(v_1, v_2)$, connecting z with v_1 and v_2. Then $(w, z_1, \ldots, z, \ldots, z_2, w)$ is a geodesic loop, again in contradiction with total convexity of M. Thus, any vertex $z \in P$ is adjacent to any neighbour of v in the set M. From the choice of v we obtain the converse inclusion. Therefore, P is a module in the graph $G(M \cup conv(v))$. Since M is totally convex, this module is simplicial.

Next we prove that the set $M \cup conv(v)$ is totally convex. Assume the contrary. Then there exists two vertices $v_1 \in conv(v)$ and $v_2 \in M$ and a geodesic between them, whose all interior vertices are outside $M \cup conv(v)$. We assert that vertices v_1 and v_2 are nonadjacent. Suppose the contrary, and let z_1 be the furthest from v_1 neighbour of v_2 in the geodesic $L(v_1, v_2)$. Then $(v_2, z_1, \ldots, v_2)$ is a geodesic loop, contrary to total convexity of M. Since P is a module, the vertex v_2 is not adjacent to any vertex of P.

Let z be the neighbour of v_2 in the geodesic $L(v_1, v_2)$. We claim that any neighbour w of v in the set M is adjacent to z, too. Indeed, as we already showed, the vertices v_1 and w are adjacent. Since M is totally convex, the vertex w is adjacent to all vertices of $L(v_1, v_2)$. In particular, w and z are adjacent. This, however, violates the initial assumption that v has the largest neighbourhood in M. This final contradiction proves the total convexity of $M \cup conv(v)$, thus completing the proof. □

Lemma 14. *Let v be the vertex of G selected in steps* (3) *and* (4) *of the algorithm MMCS. Then the set $P_i = conv(v) \setminus M$ defined in step* (6) *induces a cell of G.*

Proof. By Lemma 13 both sets M and $M \cup conv(v)$ are totally convex. If P_i is not a cell, then according to Corollary 1, P_i can be represented as a union of some cells $C_{i_1}, \ldots, C_{i_k}$ of G. Let f be a peakless function on $G(M \cup conv(v))$ which separates these cells and the set M. We can assume, without loss of generality, that f takes a constant value α_0 on M. Suppose also that $C_{i_j} = \alpha_j$, $0 < j \leqslant k$, where $\alpha_0 < \alpha_1 < \cdots < \alpha_k$. Since $M \cup C_{i_1} = [f \leqslant \alpha_1]$ is totally convex and $C_{i_1} = conv(v') \setminus M$ for any vertex $v' \in C_{i_1}$, we obtain a contradiction with the choice in step (4) of the vertex v. □

Proof of Theorem 5. Let $\mathscr{P} = (P_i)$, $0 < i \leqslant n$, be the partition of G obtained by the algorithm MMCS. Proceed by induction on n. By Lemmas 13 and 14 we conclude that the sets P_i are simplicial modules in the subgraphs $G(\cup_{j \leqslant i} P_j)$, and all they represent cells of G. By Lemma 12 we have that $\mathscr{P}$ is a prime simplicial-modular decomposition of G.

The complexity of the algorithm mainly depends on the implementation of the step (1), because all other steps are evidently polynomial. In order to implement step (1) in polynomial time it suffices to compute the interval $I(u,v)$ between any vertices $u, v \in V$ in polynomial number of operations. For this purpose we use the following breadth first search algorithm. We process the vertices of the graph beginning with u. In the iteration i we include in *geodesic*(i) all whose vertices of G which can be connected with u by a geodesic of length i. For this we maintain an array of sets *in*(x), $x \in V$. We story in *in*(x) all vertices which are neighbours of x in geodesics connecting u and x. A vertex x is included in the set *geodesic*(i) if there exists a vertex $y \in geodesic(i-1)$ adjacent to x and a vertex $z \in in(y) \cap geodesic(i-2)$ nonadjacent to x. In this case we include the vertex y in *in*(x). The maximal length of a geodesic in the graph G is $2|V|-4$ (the extremal case arises when G is a cycle of length four with a pendant path of length $|V|-4$). Therefore, we need at most $2|V|-4$ iterations. To construct the interval $I(u,v)$ we start with the vertex v and include in $I(u,v)$ all vertices of the set *in*(v). In a similar way, in the assumption that the vertex x is already included in $I(u,v)$, at the next iteration we include in $I(u,v)$ all vertices of *in*(x). Clearly, the complexity of this algorithm is polynomial. This final remark concludes the proof of Theorem 5. □

References

[1] P. Buneman, A characterization of rigid circuit graphs, Discrete Math. 9 (1974) 205–212.

[2] H. Busemann, The Geometry of Geodesics (Academic Press, New York, 1955).

[3] H. Busemann and B.B. Phadke, Peakless and monotone functions on G-spaces, Tsukuba J. Math. 7 (1983) 105–135.

[4] V. Chepoi, Some properties of convex functions on graphs (in Russian), preprint Moldova State University (1986).

[5] P.M. Dearing, R.L. Francis and T.J. Lowe, Convex location problems on tree networks, Oper. Res. 24 (1976) 628–642.

[6] R. Diestel, Simplicial decompositions of graphs: a survey of applications, Discrete Math. 75 (1989) 121–144.

[7] R. Diestel, Graph Decompositions – A Study in Infinite Graph Theory (Oxford Univ. Press, Oxford, 1990).

[8] R. Diestel, Simplicial tree-decompositions of infinite graphs I, II, III, J. Combin. Theory B 48 (1990) 197–215; Ser. B50 (1990) 93–116, 117–128.

[9] G.A. Dirac, Simplicial decompositions of infinite graphs, in: Combinatorica, Conv. Roma 1983, Sympos. Math. 28 (1986) 159–195.

[10] M. Farber and R.E. Jamison, Convexity in graphs and hypergraphs, SIAM J. Algebraic Discrete Methods 7 (1986) 433–444.

[11] R. Halin, Simpliziale Zerfällungen beliebiger (endlicher oder unendlicher) Graphen, Math. Ann. 156 (1964) 216–225.

[12] R. Halin, Graphentheorie II (Wissenschaftliche Buchgesellschaft, Darmstadt, 1981).

[13] E. Howorka, A characterization of Ptolemaic graphs, J. Graph Theory 5 (1984) 321–323.

[14] N. Innami, A classification of Busemann G-surfaces which possess convex functions, Acta Math. 148 (1982) 15–29.

[15] B. Jamison and S. Olariu, P-components and the homogeneous decomposition of graphs, SIAM J. Discrete Math. 8 (1995) 448–463.

[16] R.E. Jamison–Waldner, A perspective on abstract convexity: classifying alignments by varietes, in: Convexity and related Combinatorial Geometry, Proceedings of the 2nd University of Oklahoma Conference (Dekker, New York, 1982) 113–150.

[17] R.H. Möhring, Algorithmic aspects of comparability graphs and interval graphs, in: I. Rival, ed., Graphs and Order (Dordrecht, Holland, 1985).

[18] R.H. Möhring and F.J. Rademacher, Substitution decomposition and connections with combinatorial optimization, Ann. Discrete Math. 19 (1984) 257–356.

[19] J.H. Muller and J. Spinrad, Incremental modular decomposition, J. Assoc. Comput. Math. 36 (1989) 1–19.

[20] D.R. Shier, Some aspects of perfect elimination orderings in chordal graphs, Discrete Appl. Math. 7 (1984) 325–331.

[21] V. Soltan, Some properties of d-convex functions, I, II (in Russian), Bull. Moldavian Acad. Sci. Ser. Math. 2 (1980) 27–31; 1 (1981) 21–26.

[22] V.P. Soltan, Introduction to the Axiomatic Theory of Convexity (in Russian) (Ştiinţa, Chişinău, 1984).

[23] V. Soltan and P. Soltan, d-Convex functions (in Russian), Soviet Math. Dokl. 249 (1979) 555–558.

[24] V. Soltan and V. Chepoi, Some classes of d-convex functions in graphs (in Russian), Soviet Math. Dokl. 273 (1983) 1314–1317.

[25] R.E. Tarjan and M. Yannakakis, Simple linear time algorithms to test chordality of graphs, test acyclicity of hypergraphs, and selectively reduce acyclic hypergraphs, SIAM J. Comput. 13 (1984) 566–579.

[26] M. van de Vel, Theory of Convex Structures (Elsevier, Amsterdam, 1993).