Journal of Geometry
Vol. 50 (1994)

0047-2468/94/020030-22$1.50+0.20/0

SEPARATION OF TWO CONVEX SETS IN CONVEXITY STRUCTURES

Dedicated to Professor N.K. Stephanidis, on the occasion of his 65 birthday

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A convexity structure satisfies the separation property S_4 if any two disjoint convex sets extend to complementary half-spaces. This property is investigated for alignment spaces, n-ary convexities, and graphs. In particular, it is proven that

a) an n-ary convexity is S_4 iff every pair of disjoint polytopes with at most n vertices can be separated by complementary half spaces,

and

b) an interval convexity is S_4 iff it satisfies the analogue of the Pasch axiom of plane geometry.

A characterization of bipartite and weakly modular spaces with S_4 convexity is given in terms of forbidden subgraphs.

1 INTRODUCTION

Separation of convex sets constitutes one of the fundamental facets of the theory of convex sets in linear spaces. In order to obtain analogous results for other types of convexities notions such as hyperplane and half space have to be defined suitably. For convexities such as convexity spaces [1,9,15,17,28], join spaces [28], metric convexity of finite-dimensional normed spaces [8,31] various separation theorems by hyperplanes were obtained. For general convexity structures, discrete, topological or metric convexities, separation is usualy expressed in terms of half spaces [19,23,24,25,26,31,34-36].

The purpose of this paper is to present criteria of separation by half spaces in various classes of convexity structures. The work has gone on for several years and this paper will serve to present the obtained results. Some of the results have appeared before in the Moldova State University press [10,11,14], others previously have stated without proof.

The definitions below are taken from Kay and Womble [25], Jamison [24], Soltan [31], and van de Vel [34,36]. The rest of the paper is organized as follows. In Section 2 we give the necessary definitions. Section 3 presents the separation theorems for alignments and n-ary convexities, Section 4 for interval convexities. In Sections 5 and 6 we study the S_4 graph convexities; in particular, we characterize the bipartite spaces and weakly modular spaces with S_4 convexity.

2 BASIC NOTIONS

A *convex structure* consists of a set X together with a collection $\mathcal{C}$ of subsets of X, henceforth called *convex sets*, such that

(a) the empty set and the set X are convex;
(b) the intersection of convex sets is convex.

We often identify the pair $(X,\mathcal{C})$ with $\mathcal{C}$. If, moreover,

(c) the union of an updirected collection of convex sets is convex,

then $(X,\mathcal{C})$ is called an *alignment space* (or *alignment).*

Axiom (b) allows the construction of an associated *convex hull operator* h defined on $A \subset X$ by

$$h(A) = \cap\{C : A \subset C \in \mathcal{C}\}.$$

The hull of a finite set A is called a *polytope*, and the hull of an n–point set A is called an n*–polytope*. The points of A are called the *vertices* of a polytope $h(A)$.

A pair $(X,\mathcal{C})$ is called an n*-ary convexity structure* if

$$A \in \mathcal{C} \Leftrightarrow (\forall C \subset A, |C| \leq n \Rightarrow h(C) \subset A),$$

i.e. A is convex iff A contains any n-polytope with all the vertices from A.

Let $(X,\mathcal{C})$ be a convex structure. A convex subset of X with convex complement is called a *half space*. Let us say that two disjoint sets A, B in X are *separated* by a half space H provided $A \subset H$ and $B \subset X \backslash H$. $(X,\mathcal{C})$ is said to fulfil the *separation axiom* S_4, if any pair of disjoint convex sets in X can be separated by a half space. Axiom S_4 is usually called the *Kakutani separation property.* For sets A and B define

$$A/B = \{x \in X : h(A \cup x) \cap B \neq \emptyset\}.$$

Remark that $B \subset A/B$ and $A \subset B/A$.

3 ALIGNMENT SPACES AND n-ARY CONVEXITIES

We begin with an auxiliary result.

LEMMA 1 *For an alignment space $(X, \mathcal{C})$ the following assertions are equivalent:*
(i) $(X, \mathcal{C})$ is a S_4 convexity structure;
(ii) for any two convex sets A and B and an arbitrary point $z \in X$ if $x \in h(A \cup z)$, $y \in h(B \cup z)$, then

$$h(A \cup y) \cap h(B \cup x) \neq \emptyset;$$

(iii) for any two convex sets A and B the set A/B is convex.

PROOF. $(i) \Rightarrow (ii)$. Assume the contrary, and let $y \in h(B \cup z), x \in h(A \cup z)$ be the points for which the convex sets $h(A \cup y)$ and $h(B \cup x)$ are disjoint. Then there exist complementary half spaces H_1, H_2 of X such that $h(A \cup y) \subset H_1$ and $h(B \cup x) \subset H_2$. Suppose that $z \in H_1$. Since $x \in h(A \cup z)$ and H_1 is convex, it follows that $x \in h(A \cup z) \in H_1$, a contradiction.

$(ii) \Rightarrow (i)$. Let A_0, B_0 be disjoint convex sets in X. As $\mathcal{C}$ is an alignment, there exist maximal convex sets $A \supset A_0, B \supset B_0$ with $A \cap B = \emptyset$. We claim that $A \cup B = X$. Assume the contrary and let $z \in X \backslash (A \cup B)$. Then by maximality of the pair A, B , there exist points $x \in h(A \cup z) \cap B$ and $y \in h(B \cup z) \cap A$. Then $h(A \cup y) \cap h(B \cup x) \neq \emptyset$ whereas $h(A \cup y) = A$ and $h(B \cup x) = B$, yielding a contradiction.

$(i) \Rightarrow (iii)$. Suppose that $z \in h(A/B) \backslash (A/B)$, i.e. the convex sets $h(A \cup z)$ and B are disjoint. Let H_1, H_2 be complementary half spaces of X with $h(A \cup z) \subset H_1$ and $B \subset H_2$. Observe that $A/B \subset H_2$, and therefore $h(A/B) \subset H_2$. This yields $z \in H_1 \cap h(A/B) \subset H_1 \cap H_2$, a contradiction.

$(iii) \Rightarrow (i)$. It is sufficient to prove that any two maximal disjoint convex sets A and B are complementary half spaces. Assume the contrary, and let $z \in X \backslash (A \cup B)$. Then by maximality of the pair A, B there exist points $x \in h(B \cup z) \cap A$ and $y \in h(A \cup z) \cap B$. Hence $z \in (A/B) \cap (B/A)$. Since $A \subset B/A$ and $B \subset A/B$ we conclude that

$$y \in h(A \cup z) \subset h(B/A) = B/A, x \in h(B \cup z) \subset h(A/B) = A/B.$$

On the other hand, since $x \in A, y \in B$ and the sets A and B are convex, it follows from the definition of the sets A/B and B/A that $A \cap B \neq \emptyset$, a contradiction □

THEOREM 1 *For an alignment space $(X, \mathcal{C})$ the following assertions are equivalent:*
(i) $(X, \mathcal{C})$ is a S_4 convexity structure;
(ii) [24] any two disjoint convex polytopes can be separated by complementary half spaces;
(iii) for any two polytopes P_1, P_2 the set P_1/P_2 is convex.

PROOF. $(i) \Rightarrow (ii)$ is evident, $(ii) \Rightarrow (iii)$ is a consequence of the proof of Lemma 1.

$(iii) \Rightarrow (i)$. According to Lemma 1 it suffices to prove that for every two convex sets A and B the set A/B is convex. Let $z \in h(A/B)$. Since $\mathcal{C}$ is an alignment, there exist points $z_1, \ldots, z_k \in A/B$ such that $z \in h(z_1, \ldots, z_k)$. Let $x_i \in h(A \cup z_i) \cap B, i = 1, \ldots, k$. Further, there exist finite subsets $A_i \subset A, i = 1, \ldots, k$, such that $x_i \in h(A_i \cup z_i)$. Hence

$x_i \in h(A' \cup z_i)$, where $A' = \cup_{i=1}^{k} A_i$. Therefore, by the condition (iii) of the theorem it follows that $h(A' \cup z) \cap h(x_1, \ldots, x_k) \neq \emptyset$. We conclude that $h(A \cup z) \cap B \neq \emptyset$, showing that $z \in A/B$, as desired □

The following natural question arose. For which classes of convexities is it true that the separation property of disjoint polytopes with a prescribed numbers of vertices implies the separation property of all pairs of disjoint convex sets. For the class of all alignments the answer is negative, while for n-ary convexities we obtain the following result.

THEOREM 2 *For an n-ary convexity structure $(X, \mathcal{C})$ the following assertions are equivalent:*
(i) $(X, \mathcal{C})$ is a S_4 convex structure;
(ii) any two disjoint k-polytope and m-polytope, where $k \leq n, m \leq n$, can be separated by complementary half spaces;
(iii) for any $(n-1)$-polytope P_0 and any n-polytope P the set P_0/P is convex.

PROOF. $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are evident.

$(iii) \Rightarrow (i)$. As $\mathcal{C}$ is an alignment, then according to Theorem 1 it is sufficient to establish that for any polytopes P_1 and P_2 the set P_1/P_2 is convex. Recall, that the convex hull of any set $A \subset X$ may be constructed in the following way (consult [31] and [36]):

$$h(A) = \bigcup_{i \geq 0} h_n^i(A), \text{ where } h_n(A) = \cup\{h(B) : B \subset A \text{ with } |B| \leq n\},$$

$$h_n^0(A) = A, h_n^{i+1}(A) = h_n(h_n^i(A)), i = 1, 2, \ldots.$$

For a proof of our claim it is enough to show that $h_n(P_1/P_2) = P_1/P_2$. Choose any point $x \in h_n(P_1/P_2)$. Since $\mathcal{C}$ is an n-ary convexity, we can find n points $x_1, \ldots x_n \in P_1/P_2$, such that $x \in h_n(x_1, \ldots, x_n)$. Since for each $i = 1, \ldots, n$ we have $h(P_1 \cup x_i) \cap P_2 \neq \emptyset$ then the inclusion $x \in P_1/P_2$ is an immediate consequence of the following assertion:

for arbitrary points $y_i \in h(P_1 \cup x_i), i = 1, \ldots, n$, we have

$$h(P_1 \cup x) \cap h(y_1, \ldots, y_n) \neq \emptyset.$$

To establish this statement, we proceed by induction on $\sum_{i=1}^{n} k_i$, where $y_i \in h_n^{k_i}(P_1 \cup x_i), i = 1, \ldots, n$. The assumption is evident when $y_i \in P_1 \cup \{x_i\}, i = 1, \ldots, n$. Now, suppose that the collection of points $y_i \in h(P_1 \cup x_i), i = 1, \ldots, n$ satisfies the following conditions:

1) $y_i \in h_n^{k_i}(P_1 \cup x_i), i = 1, \ldots, n$;

2) for any collection of points $y_i' \in h_n^{m_i}(P_1 \cup x_i), i = 1, \ldots, n$, such that $\sum_{i=1}^{n} m_i < \sum_{i=1}^{n} k_i$, we have the desired property

$$h(P_1 \cup x) \cap h(y_1', \ldots y_n') \neq \emptyset.$$

At least one k_j is positive, otherwise $y_i \in P_1 \cup \{x_i\}$ for all i. Assume that $z_1, \ldots, z_n \in h_n^{k_j - 1}(P_1 \cup x_j)$ have been chosen so that $y_j \in h(z_1, \ldots z_n)$. Using our induction hypothesis, for each $i \in \{1, \ldots n\}$, select a point v_i, such that

$$v_i \in h(P_1 \cup x) \cap h(y_1, \ldots, y_{j-1}, z_i, y_{j+1}, \ldots, y_n).$$

It is evident that $h(v_1, \ldots, v_n) \subset h(P \cup x)$. So, it is enough to prove that $h(v_1, \ldots, v_n) \cap h(y_1, \ldots, y_n) \neq \emptyset$. Let $V = h(v_1, \ldots, v_n)$ and $Y = h(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)$ (y_j is omitted). By the condition (iii) of the theorem for n-polytope V and $(n-1)$-polytope Y the set Y/V is convex. Since $v_i \in h(Y \cup z_i) \cap V$, all points $z_1, \ldots, z_n$ belong to the set Y/V. Therefore for the point $y_j \in h(z_1, \ldots, z_n) \subset h(Y/V) = Y/V$ we have $h(Y \cup y_j) \cap V \neq \emptyset$, i.e. $h(v_1, \ldots, v_n) \cap h(y_1, \ldots, y_n) \neq \emptyset$. This completes the proof of the theorem □

The next example shows that the bounds in the Theorem 2 are sharp.

EXAMPLE 1 Let $X = \{x_1, \ldots x_{2n-2}, a_1, a_2, b\}$ and pick $X_0 = \{x_1, \ldots, x_{2n-2}\}$, $A = \{a_1, a_2\}$. Define

$$\mathcal{C}_1 = \{M \subset X : b \notin M\} \quad \mathcal{C}_2 = \{M_2 \subset X : b \in M, \mid M \cap X_0 \mid \leq n-2\} \cup \{M : b \in M, A \subset M\}.$$

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \{X\}$. Then $(X, \mathcal{C})$ is a convexity structure: each of the families $\mathcal{C}_1$ and $\mathcal{C}_2$ is closed under intersections, while the intersection of sets from different families $\mathcal{C}_1$ and $\mathcal{C}_2$ belongs to $\mathcal{C}_1$. For the set $M = \{x_1, \ldots x_{n-1}, b\}$ we have

$$h(M) = M \cup A \neq M = \cup\{h(M') : M' \subset M \text{ with } \mid M' \mid \leq n-1\}.$$

Therefore $\mathcal{C}$ is not an $(n-1)$-ary convexity. Now we assert that $\mathcal{C}$ is an n-ary convexity. If $b \notin M$ then $h(M) = M$, otherwise, if $b \in M$, then $h(M) \subset M \cup A$. So if $x \in h(M) \backslash M \subset A$ then $\mid M \cap X_0 \mid \geq n-1$ and for any subset $M' \subset M \cap X, \mid M' \mid = n-1$, we have $x \in h(M' \cup b)$. Hence $\mathcal{C}$ is an n-ary convexity.

Now assume that P is a polytope with at most $n-1$ vertices and M is a convex set disjoint with P. We wish to show that P and M can be separated by complementary half spaces H_1, H_2, where $P \subset H_1$ and $M \subset H_2$. If $P \cap A = \emptyset$ then put $H_1 = P$ and $H_2 = X \backslash P$. To prove the convexity of the set H_2 it suffices to remark that $H_2 \in \mathcal{C}_1$ if $b \in P$ and $H_2 \in \mathcal{C}_2$ otherwise.

Now let $P \cap A \neq \emptyset$. Then $|P \cap X_0| \leq n-2$. Hence, if $b \notin M$ we may assume that $H_1 = P \cup \{b\} \in \mathcal{C}_2$. Otherwise, if $b \in M$, then either $M \cap A = \emptyset$ and then put $H_1 = P \cup A \in \mathcal{C}_1$ or $M \cap A \neq \emptyset$ and $|M \cap X_0| \leq n-2$. In the second case let $H_2 = M$ and $H_1 = P \cup (X_0 \backslash M) \in \mathcal{C}_1$. So, in any case the sets P and M are separated by complementary half spaces. On the other hand, disjoint polytopes $P_1 = \{x_1, \ldots, x_{n-1}, a_1\}$ and $P_2 = \{x_n, x_{n+1}, \ldots, x_{2n-2}, a_2\}$ can not be separated by such half spaces. This is a consequence of next inclusions $a_1 \in h(P_2 \cup b), a_2 \in h(P_1 \cup b)$. □

Following [26,34-36], a pair $(X, \mathcal{C})$ is called a *topological convexity structure,* if X is a topological space, $\mathcal{C}$ is an alignment on X, and each polytope is closed.Two sets A, B in X are *screened* with the sets S, R of X provided that

$$S \cup R = X, A \subset X \backslash R, B \subset X \backslash S$$

A topological convex structure X is called:

a) *normal* [34] if every two disjoint convex closed sets can be screened with convex closed sets;

b) *regular* [34] (*n-regular*) if every polytope P (m-polytope P, $m \leq n$) and convex closed set C can be screened with convex closed sets.

The following result of M. van de Vel ([34], Lemma 2.1) play a fundamental role in proving the analogies of Hahn-Banach theorem for topological convexities. We give here a short proof of this proposition and some its refinements for n-ary convexities.

LEMMA 2 1)[34] *Let $\mathcal{C}$ be an alignment on X, such that every two disjoint convex sets can be screened with convex sets. Then $\mathcal{C}$ is a S_4 convexity.*
2) Let $\mathcal{C}$ be an n-ary convexity on X, such that every m-polytope and every k-polytope disjoint with it, $m, k \leq n$, can be screened with convex sets. Then $\mathcal{C}$ is a S_4 convexity.

PROOF. According to Theorems 1 and 2 it is enough to show that the set P_1/P_2 is convex if P_1 is a $(n-1)$-polytope and P_2 is a n-polytope. Assume the contrary, i.e. there is a point $z \in h(P_1/P_2)\backslash(P_1/P_2)$. Set $P = h(z \cup P_1)$. By our assumption the disjoint polytopes P_1 and P_2 can be screened with the sets S and R, where $P \subset X\backslash R, P_2 \subset X\backslash S, S \cup R = X$. So $P \subset S, P_2 \subset R$ and $P \cap R = \emptyset, P_2 \cap S = \emptyset$. Pick any $v \in P_1/P_2$. Since $P_2 \cap S = \emptyset$ and $h(P_1 \cup v) \cap P_2 \neq \emptyset$ from the convexity of sets S and R we conclude that $v \in R$. Hence $P_1/P_2 \subset R$ and therefore $z \in h(P_1/P_2) \subset R$, yielding a contradiction with $P \cap R = \emptyset$ □

THEOREM 3 *i)* [34] **(Hahn-Banach Theorem)** *Let $(X, \mathcal{C})$ be a regular convexity such that the closure of a convex set is convex.*
1) If C, D are disjoint convex sets with C closed and D compact, then there exists a closed half space $H \subset X$ with $C \cap H = \emptyset, D \subset H$.
2) If O, D are disjoint convex sets with O open and D closed, then there exists a closed half space $H \subset X$ with $O \cap H = \emptyset, D \subset H$.
ii) If $(X, \mathcal{C})$ is an n-regular n-ary convexity such that the closure of a convex set is convex, then $(X, \mathcal{C})$ satisfies 1) and 2).

THEOREM 4 *i)* [34]*Let X be compact space, and let $\mathcal{C}$ be a topological convexity on X such that the closure of a convex set is convex.Then $(X, \mathcal{C})$ is normal iff it is regular.*
ii) Let X be a compact space, and let $\mathcal{C}$ be a topological n-ary convexity on X such that the closure of a convex set is convex. Then $(X, \mathcal{C})$ is normal iff it is n-regular.

The proof of these results for n-ary convexities is the same as the proof of van de Vel results , using only Lemma 2.2) instead of Lemma 2.1).

4 INTERVAL CONVEXITIES

Let X be any (not necessarily finite) set. For each pair x, y of points in X, let xy be a subset of X, called the *interval* between x and y. Then X is an *interval space* if and only if

a) $x \in xy$ and $xy = yx$;
b) if $u \in xy$ then $uy \subset xy$,

for all $x, y \in X$. Every interval space gives rise to a convexity $\mathcal{C}$ on X :

a subset C of X is convex if and only if $xy \subset C$ for all $x, y \in C$.

Then $(X, \mathcal{C})$ is called an *interval convex structure.*

Obviously, each 2-ary convexity $\mathcal{C}$ on X is an interval convexity: it suffices to consider $xy = h(x, y)$ for all $x, y \in X$. Inversly, let $(X, \mathcal{C})$ be an interval convex structure. For any subset $A \subset X$ the convex hull $h(A)$ may be constructed in the following way:

$$h(A) = \cup_{i \geq 0} p^i(A), \quad \text{where} \quad p(A) = \cup\{xy : x, y \in A\}$$

$$p^0(A) = A, p^{i+1}(A) = p(p^i(A)), i = 1, 2, \ldots.$$

Hence, if C is any convex set of $\mathcal{C}$, then for all $x, y \in C$ we have $xy \subset C$. Thus any interval convexity $\mathcal{C}$ is an 2-ary convexity, i.e the class of the interval convexities is identical with the class of 2-ary convexities.

A particular instance of an interval space is any metric space (X, ρ): the intervals are the metric intervals

$$xy = \{z : \rho(x, z) + \rho(z, y) = \rho(x, y)\}.$$

THEOREM 5 *For an interval space X the following assertions are equivalent:*
(i) the convexity $\mathcal{C}$ on X is a S_4 convexity;
(ii) any two disjoint 2-polytopes can be separated by complementary half spaces;
(iii) (Pasch axiom) if $x \in h(a, b)$ and $y \in h(a, c)$ then $h(a, y) \cap h(b, x) \neq \emptyset$;
(iv) if $b \in h(a, x), c \in h(a, y), z \in h(x, y)$ then $h(a, z) \cap h(b, c) \neq \emptyset$.

PROOF. Implications $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$ are evident. The Pasch axiom is a particular case of the condition (iv), thus $(iv) \Rightarrow (iii)$. Further, condition $iv)$ is equivalent to the convexity of the set $a/h(b, c)$. From Theorem 2 we infer that $(i) \Leftrightarrow (iv)$.

$(iii) \Rightarrow (iv)$. By the Pasch axiom we can find a point $v \in h(x, c) \cap h(a, z)$. Repeated aplication of this axiom yields that there exists a point $w \in h(a, v) \cap h(b, c)$. Since $h(a, v) \subset h(a, z)$ we conclude that $w \in h(a, z) \cap h(b, c)$ □

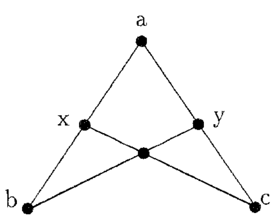

Figure 1

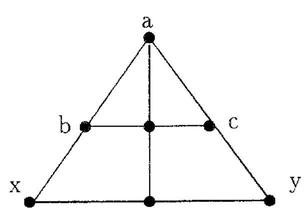

Figure 2

As a consequence of the Theorem 5 we obtain that for interval spaces the Kakutani separation property and the Pasch axiom are equivalent. This is a generalization of a well-known result of Ellis [19], which states that any convexity structure satisfying the Pasch axiom and the join-hull commutativity property is a S_4 convexity. Using this result, Ellis [19] presented a common generalization of the Kakutani separation property in real vector spaces and of the Stone theorem in distributive lattices.

Now, consider the interval analogous of the conditions (iii) and (iv) of the Theorem 5 :

$(iii)'$ (Interval Pasch axiom) if $x \in ab$ and $y \in ac$ then $yb \cap xc \neq \emptyset$;

$(iv)'$ if $b \in ax, c \in ay, z \in xy$ then $az \cap bc \neq \emptyset$.

Our next example shows that these conditions are more restrictive than the conditions of Theorem 5.

EXAMPLE 2 Let $X = \{x_i : i = 1, \ldots, 7\}$. For any points $x_i, x_j \in X$ we use the notation $(x_i x_j) = x_i x_j \setminus \{x_i, x_j\}$. Define

$$(x_1x_2) = \{x_4\}, (x_1x_3) = (x_1x_6) = (x_1x_7) = \{x_5\}$$

$$(x_2x_5) = \{x_6, x_7\}, (x_4x_5) = (x_4x_6) = (x_4x_7) = (x_6x_7) = \{x_3\},$$

and $(x_ix_j) = \emptyset$ for any other pair of points. A resulting interval convexity $\mathcal{C}$ satisfies the Pasch axiom and therefore $\mathcal{C}$ is a S_4 convexity. On the other hand, since $x_3x_4 \cap x_4x_5 = \emptyset$, then the interval Pasch axiom fails for points $x_4 \in x_1x_2$ and $x_5 \in x_1x_3$.

THEOREM 6 *For an interval space X the following implications hold:* $(iii)' \Leftrightarrow (iv)' \Rightarrow (i)$.

PROOF. The interval Pasch axiom is a particular instance of the condition $(iv)'$ and thus $(iv)' \Rightarrow (iii)'$. The proof of the implication $(iii)' \Rightarrow (iv)'$ coincide with the proof of implication $(iii) \Rightarrow (iv)$.

$(iv)' \Rightarrow (i)$. In view of Theorem 2 it is enough to show that for any points $x_1, x_2, x_3 \in X$ the set $x_1/h(x_2, x_3)$ is convex. Observe that this assertion is a consequence of the following claim:

if $b \in h(a,x), c \in h(a,y)$ and $z \in xy$ then $h(a,z) \cap bc \neq \emptyset$.

Assume that $b \in p^k(a,x), c \in p^r(a,y)$, where

$$p^{i+1}(A) = p(p^i(A)) \text{ and } p(A) = \cup\{uv : u, v \in A\}.$$

For a proof of this claim we proceed by induction on $k+r$. If $k = r = 0$ then $b \in ax, c \in ay$. By invoking the condition $(iv)'$ we obtain that $az \cap bc \neq \emptyset$. Now suppose that $k + r > 0$ and let, for example, $k > 0$. Then there exist points $b_1, b_2 \in p^{k-1}(a,x)$, such that $b \in b_1b_2$. By the induction hypothesis we can find points $v_1 \in h(a,z) \cap cb_1$ and $v_2 \in h(a,z) \cap cb_2$. Then, by virtue of $(iv)'$, there exists a point $v \in cb \cap v_1v_2$. Observe that $v \in v_1v_2 \subset h(a,z)$ and therefore $v \in cb \cap h(a,z)$ □

A convex structure $(X, \mathcal{C})$ is called *binary* [26,34-36], if each finite collection of pairwise intersecting convex sets has a nonempty intersection. The binary convexities may be defined also as convexities, satisfying the following property:

for any points $a, b, c \in X$ the set $h(a,b) \cap h(b,c) \cap h(c,a)$ is nonempty.

Binary convexities, and in special, binary convexities on topological spaces, have been studied by M. van de Vel [34-36].

Following [4,22], a space X endowed with a binary convexity $\mathcal{C}$ is called a *median space* if for any $a, b, c \in X$ the set $h(a,b) \cap h(b,c) \cap h(c,a)$ has exactly one point, which is denoted by $m(a,b,c)$. Recall that convex structure has the property S_1 if all singletons are convex.

As an example of application of our Theorem 5 we present a characterization of S_1 binary convexities, satisfying the separation property S_4.

COROLLARY 1 [35] *Let $(X, \mathcal{C})$ be a S_1 binary convexity. Then $\mathcal{C}$ is S_4 if and only if X is a median space.*

PROOF. If $x_1, x_2 \in h(a,b) \cap h(c,a), x_1 \neq x_2$, then the convex sets $\{x_1\}$ and $\{x_2\}$ can not be separated by complementary half spaces. Conversely, let X be a median space. As any median space is an interval space, then it is enough to verify that the Pasch axiom is fulfilled. Assume the contrary and choose points $a, b, c \in X, x \in h(a,b), y \in h(a,c)$ such that $h(x,c) \cap h(y,b) = \emptyset$. Denote by x_0 and y_0 the medians $m(a,x,c)$ and $m(a,y,b)$. As $x_0 \in h(a,x) \subset h(a,b), y_0 \in h(a,y) \subset h(a,c)$ and $h(x_0,c) \subset h(x,c), h(y_0,b) \subset h(y,c)$, then, without loss of generality, we may assume that $x = x_0, y = y_0$. Thus $x, y \in h(a,b) \cap h(a,c)$. Put $m_1 = m(x,b,c), m_2 = m(y,b,c)$. Since $h(x,c) \cup h(y,b) \subset h(a,b)$ and $h(x,c) \cup h(y,c) \subset h(a,c)$ we conclude that $m_1, m_2 \in m(a,b,c)$, yielding a contradiction □

5 BIPARTITE SPACES

Let X be an interval space. An interval uv is called an *edge* if $u = v$ and $uv = \{u,v\}$; the edges then form the *graph* G of the interval space X. In order to ensure that the graph G of a space X is connected some additional conditions are necessary. First, call X *geometric* [36] (or *orderable* [3,21]) if for any three points x, y, z with $y \in xz$, there exists a partial order $\leq$ on X such that $x \leq y \leq z$ and, for $x \leq z, y \in xz$ is equivalent to $x \leq y \leq z$. A partial instance of a geometric interval space is any metric space (X, ρ).

A *chain* R in an interval space X is a totally orderable subspace, that is, there exists a partial order $\leq$ on R such that for $u, v, w \in R$ one has $v \in uw$ if and only if $u \leq v \leq w$. If R admits a least element a and a largest element b, then R is called *bounded* [3]. Now, X is said to be *discrete* [3] if all bounded chains in X are finite. As was shown in [3] the graph G of a discrete geometric interval space is connected. The graph G can be regarded as a metric space, where the metric d accounts for the lengths of shortest paths. The corresponding intervals

$$I(u,v) = \{x : x \text{ is on a shortest path between } u \text{ and } v\}$$

have to be distinguished from the intervals uv of the given interval space. Now, call an interval space X *graphic* [3] if the equality $uv = I(u,v)$ holds for all u, v of X.

For graphic spaces the results of the previous sections may be refined in the following way.

THEOREM 7 *For a graphic space X the following assertions are equivalent:*
(i) convexity $\mathcal{C}$ on X is S_4;
(ii) any two disjoint intervals ab and cd can be separated by complementary half spaces;
(iii) X satisfies the interval Pasch axiom.

PROOF. First, note that the implication $(ii) \Rightarrow (iii)$ is trivial and that $(iii) \Rightarrow (i)$ is a part of Theorem 6.
$(i) \Rightarrow (ii)$. It is sufficient to verify that in a graphic space X with a S_4 convexity any interval uv is convex. Assume the contrary, and let xy be a non-convex interval for which the distance $d(x,y)$ in the graph G is as small as possible. Let $z_1z_2 \subset xy$ for some points $z_1, z_2 \in xy$. Without loss of generality, assume that these points z_1, z_2 are chosen in xy at the minimal distance too. Then $v \in xy$ for any point $v \in z_1z_2, v \neq z_1, z_2$. Let z be any neighbour of $z_1 \in z_1x$. Since $z \notin xy$ and X is graphic, $z_1 \in xz \cup yz$. Assume, for example, that $z_1 \in zx$. Since $d(z_1, y) < d(x, y)$ in G, the interval yz_1 is convex. Consider any disjoint half spaces H_1 and H_2, separating the convex sets yz_1 and $\{z\}$: let $yz_1 \subset H_1$ and $z \in H_2$. If $x \in H_2$, then $z_1 \in zx$, yielding a contradiction. Therefore $x \in H_1$. However in this case $x, y \in H_1$ and $z \in h(x,y) \subset H_1$, which is impossible □

An interval space X is said to satisfy the *triangle condition* [3,13] if for any three points u, v, w in X with

$$uv \cap uw = \{u\}, vu \cap vw = \{v\}, wv \cap wu = \{w\},$$

the intervals uv, uw, vw are edges whenever at least one of them is an edge.

THEOREM 8 [3] *Any discrete geometric interval space X satisfying the triangle condition is graphic.*

Let X be a discrete orderable interval space. For any edge uv set

$$W(u,v) = \{x \in X : u \in xv\}, W(v,u) = \{x \in X : v \in xu\}$$

Then call the discrete orderable interval space X *bipartite* if for any edge uv of this space we have $W(u,v) \cup W(v,u) = X$. From the Theorem 8 we conclude that any bipartite space is graphic. Next we present a recursive characterization of bipartite spaces, whose convexities are S_4.

Recall that a *hypercube* is the undirected Hasse diagram of the lattice of all finite subsets of some set. A subgraph H of the graph G is an *isometric subgraph* of G if the distance $d_H(u,v)$ between any two points u and v in H equals their distance $d_G(u,v)$ in the larger graph G. A graph H can be *embedded isometrically* into a graph G if H may be represented as an isometric subgraph of G.

In addition to S_4 , we consider the following separation properties (cf. [24],[31],[36]):

S_2: Any two distinct points are in complementary half-spaces.

S_3: Any convex set is an intersection of half-spaces.

The following proposition is essentialy based on a Djokovic [16,30] characterization of isometric subgraphs of hypercubes.

LEMMA 3 [2,11] *The convexity C of a bipartite space X is S_3 (or S_2 respectively) if and only if the graph G of this space is isometric embeddable into a hypercube.*

In order to characterize median graphs M. Mulder [27] introduced the concept of a convex expansion of a graph. A similar construction was introduced in [12], with the purpose to characterize isometric subgraphs of hypercubes and Hamming graphs.

Let $G_0 = (X_0, E_0)$ be a connected graph, and let W_1^0 and W_2^0 be two subsets of X_0 such that

$$1) W_1^0 \cap W_2^0 \neq \emptyset; \quad 2) W_1^0 \cup W_2^0 = X_0; \quad 3) \{(x,y) \in E_0 : x \in W_1^0 \backslash W_2^0, y \in W_2^0 \backslash W_1^0\} = \emptyset.$$

4) subgraphs $G_0(W_1^0)$ and $G_0(W_2^0)$, induced by sets W_1^0 and W_2^0, are isometric subgraphs of the graph G_0.

An *isometric expansion* of G_0 with respect to W_1^0 and W_2^0 is a graph $G = (X, E)$ constructed in the following way from G_0 :

(i) replace each point $x \in W_1^0 \cap W_2^0$ by two points x_1 and x_2 which are joined by an edge;

(ii) join x_1 to all neighbours of x in $W_1^0 \backslash W_2^0$ and join x_2 to all neighbours of x in $W_2^0 \backslash W_1^0$;

(iii) if $x, y \in W_1^0 \cap W_2^0$ are adjacent in G_0, then join x_1 to y_1 and x_2 to y_2.

Let W_1 be the subset of G which consists of $W_1^0 \backslash W_2^0$ together with the "first" copy x_1 of each point $x \in W_1^0 \cap W_2^0$. The set W_2 is described similarly. We illustrate this construction in Fig.3.

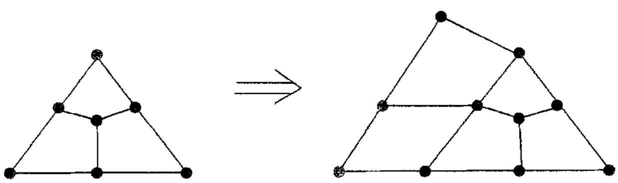

Figure 3

The isometric expansion of a graph G_0 with respect to the sets W_1^0 and W_2^0 can be defined as a multivalued mapping $\psi : X_0 \to X$:

for any $x \in X_0$ define $\psi(x) = \{x_1, x_2\}$ if $x \in W_1^0 \cap W_2^0$ and $\psi(x) = \{x\}$ otherwise.

For any subset $A_0 \subset X_0$ put

$$\psi(A_0) = \{x \in X : x \in \psi(x_0), x_0 \in A_0\}.$$

It is not difficult to observe that the sets W_1 and W_2 are complementary half spaces of the graph G. The inverse mapping $\lambda = \psi^{-1}$ is called an *isometric contraction* of the graph G with respect to the half spaces W_1 and W_2. Since the graph G is bipartite, any point of W_1 is adjacent to at most one point of W_2, and viceversa. Hence, the graph G_0 is obtained as a result of identification of all pairs of adjacent points (x_1, x_2), where $x_1 \in W_1, x_2 \in W_2$. For any point $z \in X$ set $\lambda(z) = \{z_0\}$, where $z \in \psi(z_0)$. For a subset $A \subset X$ denote $\lambda(A) = \cup\{\lambda(z) : z \in A\}$.
Let us denote by $I(G_0)$ the class of all graphs, which can be obtained from the graph G_0 by a sequence of isometric expansions. Let K_1 be an one-point graph.

THEOREM 9 [12] *A graph G is isometric embeddable into a hypercube if and only if the convex hull $h(A)$ of any finite set A is finite and $G(h(A)) \in I(K_1)$. In particular, a finite graph G is isometric embeddable into a hypercube if and only if $G \in I(K_1)$.*

By Theorem 9 any bipartite graph G with a S_4 convexity is isometric embeddable into a hypercube. Then each convex set of G is the intersection of a subcube with G. In particular, each pair of complementary half spaces of G is generated by a pair of complementary maximal proper subcubes. Therefore any isometric contraction of G is equivalent to collapsing one dimension of a hypercube in which G is embedded.

A S_3 convexity of a bipartite space is not necessarily S_4. So, there are five isometric subgraphs of the 4-dimensional cube, the convexity of which is not S_4 (Fig.4). Denote these graphs by L_1, L_2, L_3, L_4, L_5 .

THEOREM 10 *A convexity C of a finite bipartite space X is S_4 if and only if the graph of this space satisfies the following conditions:*
a) $G \in I(K_1)\backslash I(L_i)$;
b) if $A \in C$ then $G(A) \neq L_i, i = 1, \ldots, 5$.

In general, a convexity of a bipartite space X is S_4 if and only if the graph G of this space is isometric embeddable into a hypercube and for each finite convex set $A \subset X$ we have $G(A) \notin \cup_{i=1}^{5} I(L_i)$.

The proof of this theorem amounts to the following properties of isometric expansions of graphs.

LEMMA 4 *If $x \in W_1, y \in W_2$ and $z_0 \in x_0y_0$ in G_0 then $\psi(z_0) \subset xy$ in G.*

PROOF. Assume that $z \in \psi(z_0) \cap W_1$. Then

$$d_G(x,z) = d_{G_0}(x_0, z_0), d_G(z,y) = d_{G_0}(z_0, y_0) + 1, d_G(x,y) = d_{G_0}(x_0, y_0) + 1,$$

and therefore $z \in xy$ in G □

LEMMA 5 *For any pair of points $x_0, y_0 \in X_0$ there exist two points $x \in \psi(x_0), y \in \psi(y_0)$ such that $xy = \psi(x_0y_0)$, where xy is an interval of G and x_0y_0 is an interval of the graph G_0. Inversly, for any points x, y of G we have $x_0y_0 = \lambda(xy)$, where $x_0 = \lambda(x)$ and $y_0 = \lambda(y)$.*

PROOF. First, assume that the sets $\psi(x_0)$ and $\psi(y_0)$ belong to the same half space. Then for each point $z_0 \in x_0y_0$ we have $|\psi(z_0)| = 1$. Therefore if $x = \psi(x_0), y = \psi(y_0), z = \psi(z_0)$ then

$$d_G(x,z) = d_{G_0}(x_0,z_0), d_G(z,y) = d_{G_0}(z_0,y_0), d_G(x,y) = d_{G_0}(x_0,y_0),$$

and so $z \in xy$. Next suppose that $x_1 \in \psi(x_0) \cap W_1, y_2 \in \psi(y_0) \cap W_2$. Using the preceding lemma, we obtain that for any point $z_0 \in x_0y_0$ we have $\psi(z_0) \subset x_1y_2$. Hence $x_1y_2 = \psi(x_0y_0)$ □

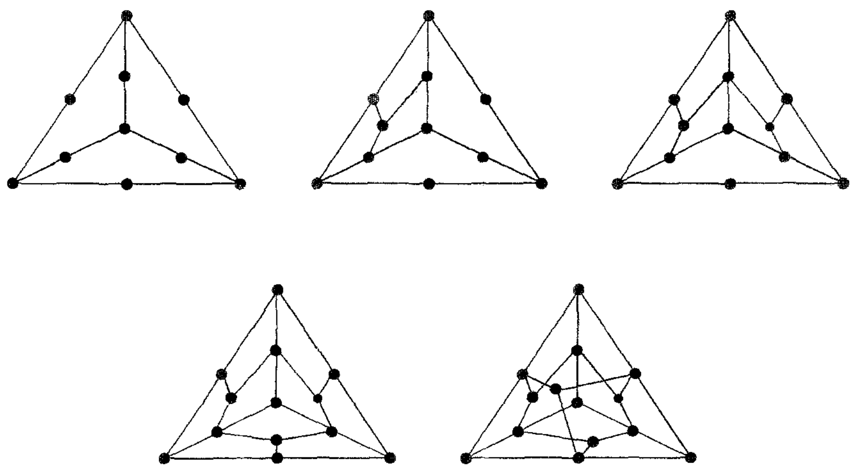

Figure 4

LEMMA 6 *If a graph G is isometric embeddable into a hypercube Q_α then any interval xy of G is convex.*

PROOF. Any interval $I = xy$ of a hypercube Q_α induces an n-cube [27], where $n = d(x,y)$. This interval is convex in Q_α. Now observe that the interval xy of the graph G coincide with the set $I \cap X$ and thus xy is convex in G □

Observe that, for any edge $e = (x,y)$ of an isometric subgraph G of a hypercube, the sets $W(x,y), W(y,x)$ are complementary half spaces. In what follows we will use the short-hand $G_e = \lambda_e(G)$, where G_e is a graph obtained by an isometric contraction of the graph G with respect to the sets $W(x,y)$ and $W(y,x)$. Put also $G = \psi_e(G_e)$.

PROOF of the THEOREM 10. Convexity in any graph $L_i, i = 1,\dots,5$, is not S_4 . Therefore any bipartite space with S_4 convexity satisfies condition b) of the theorem. Now, suppose that a graph G is obtained from the graph $G_0 \in I(L_i)$ by an isometric expansion ψ. We will show that if the convexity of X_0 is not S_4, then the convexity of X is not S_4 too. According to Theorem 7, there exist points $a_0, b_0, c_0 \in X_0$ and $x_0 \in a_0c_0, y_0 \in b_0c_0$ such that $b_0x_0 \cap a_0y_0 = \emptyset$. By the reference to Lemma 5 we are able to choose the points

$$a \in \psi(a_0), b \in \psi(b_0), c, c' \in \psi(c_0), x \in ac \cap \psi(x_0), y \in bc' \cap \psi(y_0).$$

Then, by virtue of Lemma 4, at least one the relations $x \in ac'$ or $y \in bc$ is valid. Let, for example, $y \in bc$. First assume that there exists a point $z \in ay \cap bx$. Then by Lemma 5 we have $\lambda(z) \in a_0y_0 \cap b_0x_0$, in contradiction with the choice of points a_0, b_0, c_0, x_0, y_0. So $ay \cap bx = \emptyset$ and therefore the convexity of X is not S_4. Thus any bipartite space X with S_4 convexity satisfies all conditions of the theorem .

Converslу, assume that the graph G of a finite bipartite space X satisfies condition b) of the theorem and is not S_4. Before starting this part of the proof, we assume that the graph $G \in I(K_1)$ is maximally contracted to a non-S_4 space.

Among all triples of points, which fail the condition *iii*) of the Theorem 7, we choose a triple $A = \{a, b, c\}$ for which the perimeter $p = d(a, b) + d(b, c) + d(c, a)$ is as small as possible. Then there exist points $x \in ab, y \in ac$ such that $xc \cap yb = \emptyset$. We claim that points x and y may be considered adjacent to a. Assume the contrary, and choose x and y in such a way, that the sum $d(x, a) + d(y, a)$ is minimal. Let $v \in ay$ be a neighbour of a. Since $v \neq y$, we conclude that $vb \cap xc \neq \emptyset$. By z we denote a point of this intersection. If $v \in ab$ then the set A may be replaced by a triple $\{v, b, c\}$ of perimeter $p - 2$, which fail the interval Pasch axiom too. Thus $a, x \in vb$. Therefore $x, b \in W(a, v), y, c \in W(v, a)$. Set $e = (v, a)$. By Lemma 5 we conclude that $x_0 \in a_0b_0, y_0 \in a_0c_0$ and moreover $x_0 \neq a_0, b_0$ and $y_0 \neq a_0, c_0$. As the convexity of X_0 is S_4, then there is a point $w_0 \in x_0c_0 \cap y_0b_0$. By Lemma 4 we have $\psi_e(w_0) \subset xc \cap yb$, yielding a contradiction. Hence, the points x and y are adjacent to a, and moreover $x, y \in ab \cap ac$.

Next we wish to prove that $d(b, x) = d(y, c) = 2$. Assume the contrary and let $d(b, x) \geq d(c, y)$. First, consider the case when $d(c, y) = 1$. Since $x \in ac$ and $d(a, c) = 2$ we conclude that the point x is a neighbour of c. Necessarily, both points x and y belong to the interval bc. Otherwise, if $y \notin bc$ then $c \in yb \cap xc$, which is impossible. If $x \notin bc$ then $c \in xb \subseteq ab$ since x is a neighbour of c. Hence, $y \in ac \cap cb = c$, a contradiction. If $x, y \in bc$ we have $a \in xy \subseteq bc$ (cf. Lemma 6), whence $a \in by \cap cx$. So, $d(x, b) \geq d(y, c) \geq 2$ and $d(x, b) > 2$. Pick any point $v \in xb$ adjacent to x and let $e = (x, v)$. By the initial assumption the convexity of a graph $G_e = \lambda_e(G)$ is S_4. Thus for given points $x_0 \in a_0b_0$ and $y_0 \in a_0c_0$ there is a point $z_0 \in x_0c_0 \cap y_0b_0$. Suppose that $c \in W(v, x)$. Then $y \in W(x, v)$, for otherwise the points v and y will be adjacent and $v \in xc \cap yb$. Therefore $x, y \in W(x, v), b, c \in W(v, x)$. By Lemma 4 $\psi_e(z_0) \subset by \cap cx$, yielding a contradiction. Hence $c \in W(x, v)$. As the interval ac is convex, then $d(v, y) \geq 2$ and so $y \in W(x, v)$. By Lemma 3 $\psi_e(z_0) \subset vc \cap by$. Let p_1 be a perimeter of the triple $\{a, v, c\}$. We prove that $p_1 = p$, i.e. $b \in vc$. Obviously, $p_1 \leq p$. Let $z \in \psi_e(z_0)$. If $p_1 < p$, then by our assumption there is a point $w \in az \cap xc$. A perimeter of the triple $\{a, z, y\}$ is at most p_1 and thus $wc \cap zy \neq \emptyset$. Since $wc \subset xc$ and $zy \subset by$ we obtain a contradiction. Hence $b \in vc$ and $p_1 = p$.

Choose any point $u \in vb$ adjacent to v and set $t = (u, v)$. As we already proved, $b, c \in W(u, v)$ and $a, x \in W(v, u)$. As the convexity of a graph $G_t = \lambda_t(G)$ is S_4, then for points $x_0 \in a_0b_0, y_0 \in a_0c_0$ there exists a point $z_0 \in x_0c_0 \cap y_0b_0$. If $y \in W(v, u)$, then by Lemma 4 we have $\psi_t(z_0) \subset by \cap cx$, a contradiction. Therefore $y \in W(u, v)$, so that $d(u, y) = 2$ and $u \in yb$. Hence $uy \cap xc = \emptyset$ and the point b may be replaced by a new point u. So, we can assume that $d(x,b) = 2 = d(y,c)$. Then $d(a,b) = d(a,c) = 3$ and we conclude that $d(x,c) = 2 = d(y,b)$. As $a \notin bc$ then $d(b,c) \leq 3$, otherwise $x, y \in bc, a \in xy \subseteq bc$ (cf. Lemma 6), whence $a \in by \cap cx$. If $d(b,c) = 3$ or $d(b,c) = 1$ we obtain an odd cycle of the graph G. Hence $d(b,c) = 2, d(a,b) = d(a,c) = 3$. Let

$$z_1 \in xb, z_2 \in yc, u_1 \in xc, u_2 \in yb, u \in bc.$$

Put $S = \{a, b, c, x, y, z_1, z_2, u_1, u_2, u\}$. First we show that $d(u,a) = 4$. Otherwise, if $d(u,a) = 2$ then $u, x, y \in ab \cap ac$. By the convexity of intervals ab and ac we conclude that u is adjacent to x and y. But then $u \in by \cap cx$, in contradiction with the choice of points a, b, c, x and y. Thus any common neighbour of points c and b is at distance four from a. In particular, $d(a,u) = 4$ and $c, b \in au$. Hence G consists of the interval au only. This is so, because the cube in which G is isometrically embedded, can be further contracted to a four-dimensional subcube containing the interval au. By assumption, however, G has been maximally contracted, so it is already embedded in four dimensions.

Next we claim that points of the set S are distinct. It suffices to prove this only for points z_1, z_2, u_1 and u_2. As G is bipartite, we deduce that $u_1 \neq u_2$. Since each common neighbor of b, c is at a distance 4 from a we get $z_1 \neq u_1$ and $z_2 \neq u_2$. Since a is at distance four from any common neighbour of points b and c then $z_1 \neq z_2$. Finally suppose that points y and z_1 are adjacent. Then $z_1 \in xy$. As the interval ac is convex and $x, y \in ac$ then points c and z_1 must be adjacent, which is impossible. Hence, $z_1 \neq u_2$. That $z_2 \neq u_1$ follows for reason of symmetry.

Thus all points of S are distinct. As G is bipartite and all distances between points of S, except $d(a,u)$, are at most three then $G(S)$ is an isometric subgraph of the graph G. Moreover, $G(S) = L_1$. Since graph G is an isometric subgraph of the four-dimensional cube then the straightforward verification shows that G coincides with one of the graphs $L_1, \ldots, L_5$. However, in this case condition b) of the theorem would be violated.

The second part of the theorem is an immediate consequence of Theorem 7, of the first part of this theorem and of the finiteness of the hull of any finite set $A \subset X$ (Theorem 9). This completes the proof of the theorem □

EXAMPLE 3 Figure 5 provides an example of a bipartite space which satisfies condition a) and fails condition b) of Theorem 10. The convexity of this space is not S_4.

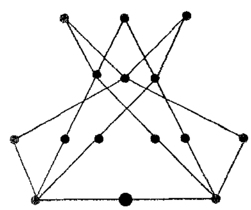

Figure 5

6. WEAKLY MODULAR SPACES

An interval space X is said to satisfy the *quadrangle condition* if $v, w \in ux$ and $x \in vw$ such that vx and wx are edges implies that $uv \cap uw \cap vw$ contains a point y such that vy and wy are edges. Following [3], call a discrete orderable interval space X *weakly modular* if it satisfies the triangle and quadrangle conditions (Fig.6-7).

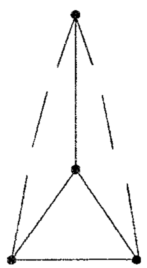

Figure 6

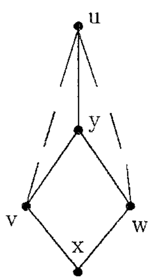

Figure 7

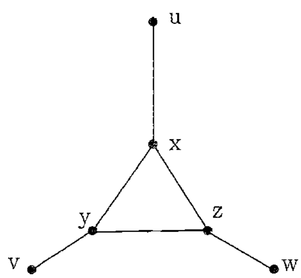

Figure 8

In view of the Theorem 8, weakly modular spaces are graphical. Weakly modular graphs were previously studied in [7] and [13]. Particular subclasses are formed by pseudo-modular graphs (Bandelt and Mulder [5,6]), chordal graphs (Dirac [18]), bridged graphs (Farber and Jamison [20] and Soltan and Chepoi [32]) and Helly graphs (Quilliot [29]).

Points x, y, z of an interval space X form a *metric triangle* (xyz) [3,13] if the following interval conditions are satisfied:

$$xy \cap xz = \{x\}, yx \cap yz = \{y\}, zx \cap zy = \{z\}.$$

A metric triangle (xyz) is *equilateral* if $d(x,y) = d(x,z) = d(y,z) = k$ in the graph G of this space. The number k is called the *size* of the metric triangle. Let u, v, w be three points of X. Then x, y, z form a *pseudo-median of size* k of the triple u, v, w if (xyz) is an equilateral metric triangle of size k and the following properties are satisfied (Fig.8) :

$$x, y \in uv \quad ; \quad x, z \in uw \quad ; \quad y, z \in vw$$

and

$$x \in uy \cap uz \; ; \; y \in vx \cap vz \; ; \; z \in wx \cap wy \; .$$

Following [3], call a space X *weakly median* if X is weakly modular and the graph G of this space does not contains any one of the graphs of Figure 9 as an induced subgraph.

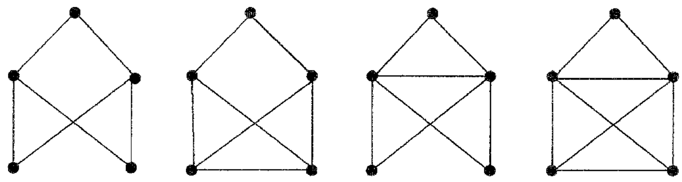

Figure 9

THEOREM 11 *A convexity $\mathcal{C}$ of a weakly modular space X is S_4 if and only if X is a weakly median space.*

We commence by establishing a number of auxiliary results.

LEMMA 7 [13] *Let X be a weakly modular space. Then*
i) each metric triangle of X is equilateral ;
ii) if (xyz) is a metric triangle of size k then $d(x,v) = k$ for each $v \in yz$;
iii) if $xu \cap yz = \{y\}$ and $xv \cap yz = \{z\}$ then $d(u,v) \geq d(y,z)$.

Let a set $A \subset X$ induce a connected subgraph in the graph G of a space X. Call the set A *locally convex* if $x, y \in A, z \in xy$ and z is a common neighbour for x and y, implies that $z \in A$.

LEMMA 8 *In a weakly modular space X any locally convex set $A \subset X$ is convex.*

PROOF. For any points $x, y \in A$ by $l(x,y)$ denote the length of a shortest (x,y)-path L inside A. Obviously $l(x,y) \geq d(x,y)$. In order to prove that $xy \subset A$ we proceed by induction on $l(x,y)$. Since A is a locally convex set then $xy \subset A$ if $l(x,y) \leq 2$. Now assume that $ab \subset A$ for any $a, b \in A$ with $l(a,b) < n$.

Let $x, y \in A, l(x,y) = n$ and choose any point $v \in xy$. Denote by z a neighbour for x in the path L. Let w be a neighbour for x in the interval xv. By induction hypothesis we have $zy \subset A$. If $d(z,y) > d(x,y)$ then $v \in xy \subset zy \subset A$. Thus $d(z,y) \leq d(x,y)$.

If $d(z,y) = d(x,y) - 1$ then the points w and z are equidistant from y. By the quadrangle condition there is a point $u \in wy \cap zy$ adjacent to w and z. As $u \in zy \subset A$ and set A is locally convex we conclude that $w \in xu \subset A$. Since $l(w,y) < n$ by the induction assumption it follows that $v \in wy \subset A$.

We next consider the case when $d(z,y) = d(x,y)$, i.e. L is a path of length $d(x,y) + 1$. By the triangle condition there exists a common neighbour $u \in zy \cap xy$ of the points z and x. Then $u \in zy \subset A$ and there exists a (x,y)-path $L_0 \subset A$ of length $d(x,y)$. This path L_0 consists of the edge (x,u) and some shortest path between u and y. Thus, we arrive at a final contradiction □

LEMMA 9 *For any two points x, y of a weakly median space X the set $x/y = \{z \in X : y \in xz\}$ is convex.*

PROOF. We proceed by induction on $d(x,y)$. First assume that x and y are adjacent. If $z \in x/y$ then $yz \subset x/y$ and thus the set x/y induces a connected subgraph of a graph G. By Lemma 8 it suffices to show that the set x/y is locally convex.

Let $z_1, z_2 \in x/y, d(z_1,z_2) = 2$ and $z \in z_1z_2$. We prove the inclusion $z \in x/y$ by induction on $p = d(z_1,y) + d(z_2,y)$. If $p = 2$ then either $z_1 = y$ or y is adjacent to z_1 and z_2. Either case forces $y \in xz_2, z \in yz_2$ and thus $y \in xz_2$. In the second case $y \notin xz$. This implies that either x and z are adjacent and we get a forbidden graph, or $d(x,z) = d(y,z) = 2$. By the triangle condition there exists a point w, adjacent to x,y,z. Then w must be adjacent to z_1 and z_2, otherwise the points y,z,z_1,z_2,w induce a forbidden graph. But in this case points x,y,w,z_1,z_2 induce a forbidden graph too.

Next assume that $d(z_1,y) + d(z_2,y) = p > 2$ and let $d(x,z_2) \geq d(x,z_1)$. If $d(x,z_2) > d(x,z_1) + 1$ then $z_1z_2 \subset z_1y \cup z_2y$ and so $z \in x/y$. Hence $d(x,z_2) \leq d(x,z_1) + 1$. Moreover, $d(z,x) \leq \min\{d(z_1,x), d(z_2,y)\}$, for otherwise at once $z \in x/y$. We distinguish two cases.

Case 1. $d(x,z_1) = d(x,z_2) = k$.

Then $d(y,z_1) = d(y,z_2) = k-1$ and the size of the pseudo-median of the triple y,z_1,z_2 is 0 or 2. In the second case this pseudo-median is the metric triangle $(y_0z_1z_2)$, where $d(y_0,z_1) = d(y_0,z_2) = 2$. By the triangle condition and Lemma 6, ii) we deduce that $d(y_0,z) = 2$ and therefore there is a point $u \in y_0z_1 \cap y_0z$ adjacent to y_0,z_1 and z. Since $(y_0z_1z_2)$ is a metric triangle, we have $d(z_2,u) = 2$.

Now suppose that the triple y,z_1,z has a pseudo-median of size 0, i.e. there is a point $s \in yz_1 \cap yz \cap z_1z_2$. In this case s is a pseudo-median of the triple x,z_1,z_2 too. Necessarily, $s = z$, for otherwise $y \in xz$ and $z \in x/y$. So $d(x,z) = a$ and points z and s are non-adjacent, otherwise $s \in xz$ and thus $y \in xs \subset xz$. Hence the triple x,s,z has a pseudo-median x_0,s,z_0 of size 1. Since the point z_0 is adjacent to s and z, then z_0 must be neighbour to z_1 and z_2, otherwise the points z_1,z_2,s,z_0,z induce a forbidden graph. Further, observe that the points x_0,s,z_0,z_1,z_2 induce a forbidden graph, except the case when x_0 is adjacent to z_1 and z_2. But in this case $x_0 \in z_1z_2 \cap z_1x \cap z_2x$ and the triple x,z_1,z_2 has two distinct pseudo-medians, contradicting our assumption.

Case 2. $d(x,z_1) = d(x,z_2) - 1 = d(x,z)$.

In this case both triples x,z_1,z and y,z,z have one and the same pseudo-median y_0,z_1,v of size 1, where y_0 is adjacent to z_1 and v,and v to y_0,z_1,z_2. If $v = z$ then $y \in xz$ and $z \in x/y$. So, $v \neq z$. Since $z,v \in z_2x$ by the quadrangle condition there exists a point $w \in vx \cap zx$, adjacent to v and z. Certainly, points w and z_1,v and z must be adjacent, for otherwise the points z_1,z_2,z,v,w induce a forbidden configuration. Further, as w and z_2 are non-adjacent, then we would get one of the forbidden induced subgraphs, a contradiction. This settle the case 2.

Finally, assume that $d(x,y) > 1$ and let x_0 be any neighbor of x in the interval xy. If $z_1,z_2 \in x/y$ then $x_0 \in xy \subset xz_1 \cap xz_2$ and $y \in x_0z_1 \cap x_0z_2$. So $z_1,z_2 \in x/x_0 \cap x_0/y$. By the induction assertion the sets x/x_0 and x_0/y are convex. Thus if $z \in z_1z_2$ then $y \in x_0z, x_0 \in xz$ and hence $y \in x_0z \subset xz$, i.e. $z \in x/y$ □

LEMMA 10 *For any metric triangle (xyz) of size 1 of a weakly median space X the set $x/yz = x/y \cup x/z$ is convex.*

PROOF. The set x/yz induces a connected subgraph in G. According to Lemma 8 it suffices to prove that the set x/yz is locally convex. Since x/y and x/z are convex sets it is enough to show that $v_1v_2 \subset x/yz$ if $v_1 \in x/y, v_2 \in x/z$ and $d(v_1, v_2) = 2$. We proceed by induction on $p = d(y, v_1) + d(z, v_2)$.

If $p = 1$ then either $y = v_1$ or $z = v_2$. Assume, for example, that $y = v_1$. Then either $y \in xv$ for any $v \in v_1v_2$ or x is adjacent to any point $v \in v_1v_2$ and thus the points x, y, z, v, v_2 induce one of the forbidden graphs.

Now consider the case $p = 2$, i.e. assume that y is adjacent to v_1 and z is adjacent to v_2. Then the triple x, v_1, v_2 has a pseudo-median of size 0 or 2. In the first case there exists a point w adjacent to x, v_1 and v_2. Then w must be non-adjacent to points z and y, for otherwise one of the sets $\{x, y, z, w, v_1\}$ or $\{x, y, z, w, v_2\}$ induce a forbidden graph. By the triangle condition there is a point z_0 adjacent to v, z and v_1. If z_0 is non-adjacent to x or v_1, then points x, z, z_0, v, v_2 induce one of the first two graphs of Fig.9. However in the second case the points x, y, z, z_0, v_1 induce one of the graphs of Fig.9, thus leading to a contradiction.

Next we assume that the points x, v_1 and v_2 form a metric triangle of size 2. Let $v \in v_1v_2$. If v is non-adjacent to the points y and z, then by the triangle condition there is a point w, adjacent to z, y and v. First suppose that x and w are adjacent. Then at least one of the sets $\{v_1, y, w, x, z\}$ or $\{v_2, y, w, x, z\}$ induce a forbidden configuration unless w is non-adjacent to v_1 and v_2. Since v is non-adjacent to y and z, and y is non-adjacent to v_2, there exists a point $u \neq w$ adjacent to y, v and v_2. Then points v_1, v, w, y, u induce a graph of Fig.9, unless u is adjacent to w and v_1. Then, however, the points v_1, u, v, w, v_2 induce the forbidden graph, thus giving a contradiction.

Now we consider that x and w are non-adjacent. Then $v_1, w \in x/y$ and $v_2, w \in x/z$. By the convexity of these sets we deduce that either $v \in v_1w \cup v_2w$ and so $v \in x/y \cup x/z$ or the point w is adjacent to v_1 and v_2. In the second case we can find a common neighbour s of points x, v_1 and v. Since (xv_1v_2) is a triangle, s and v_2 are non-adjacent. If the points s and z are adjacent, then points z, s, w, v, v_1 induce a second or a last forbidden graph of Fig.9. Therefore, $d(s, z) = d(v_1, z) = 2$ and by the triangle condition we can find a point t adjacent to z, s and v_1. If $t = w$ then s and w are adjacent and the points v_1, s, w, v, v_2 induce a forbidden graph. Thus $t \neq w$. If t is adjacent to x then the points x, y, t, z, v_1 induce a graph of Fig.9. So t and x are non-adjacent. Further, if t is non-adjacent to w then $t, w \in x/z$ but $v_1 \in tw\backslash(x/z)$, yielding a contradiction with Lemma 8. On the other hand, if t and w are adjacent, then the points y, t, w, z, v_2 induce one of the last two forbidden graphs, unless t and v_2 are adjacent. In order to avoid forbidden configurations induced by points v_1, t, w, z, v_2, the point v must be adjacent to both y and z. Thus $v \in x/yz$. This settles the case $p = 2$.

Finally assume that $d(y, v_1) + d(z, v_2) = p > 2$. Denote by x_0 the furthest from x point of the intersection $xv_1 \cap xv_2$. As $\max\{d(x, v_1), d(x, v_2)\} > 2$ then $x_0 = x$. Let x_1 be a neighbour of x on the interval xx_0. The set x/x_1 is convex and $x_1 \in xv_1 \cap xv_2$ thus $x_1 \in xv$. By the quadrangle condition there exist points $y_1 \in x_1v_1 \cap yv_1$ and $z_1 \in x_1v_2 \cap zv_2$, such that y_1 is adjacent to x_1, y and z_1 is adjacent to x_1, z (Fig.10) . Since $y \in xy_1, z \in xz_1$ and $x \notin x/yz$, in virtue of case $p = 2$ we deduce that y_1 and z_1 must be adjacent. Hence $v_1 \in x_1/y_1, v_2 \in x_1/z_1$ and $d(v_1, y_1) + d(v_2, z_1) < p$. By the induction hypothesis we have $v \in x_1/y_1 \cup x_1/z_1$. Let, for example, $v \in x_1/y_1$, i.e. $y_1 \in vx_1$. As $x_1 \in xv, y_1 \in vx_1$ and $y \in xy_1$ we deduce that $y \in xv$ and thus $v \in x/yz$ □

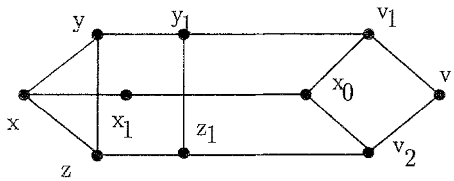

Figure 10

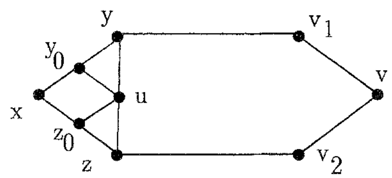

Figure 11

PROOF of the THEOREM 11. If the convexity of X is S_4, then the graphs of Figure 9 cannot occur as induced subgraphs in the graph G of this space. Indeed, the crossing edges in all diagrams can not be separated by complementary half spaces.

Conversly, assume that a space X is weakly median and we wish to show that the convexity of X is S_4. In virtue of Theorems 5 and 6 it suffices to prove that the set x/yz is convex for arbitrary three points $x, y, z \in X$. Since x/yz induce a connected subgraph, by Lemma 8 it is enough to show that the set x/yz is locally convex. Let $v_1, v_2 \in x/yz, d(v_1, v_2) = 2$ and pick any point $v \in v_1v_2$. Without loss of generality, we may assume that $xv_1 \cap yz = \{y\}, xv_2 \cap yz = \{z\}$, otherwise we can replace the points y and z by the nearest pair of points from these intersections. Assume also that $xy \cap xz = \{x\}$, otherwise for any point $x_0 \in xy \cap xz$ we have $v_1, v_2 \in x/x_0$. By Lemma 9 the set x/x_0 is convex and thus $x_0 \in xv$. As $y \in x_0v_1$ and $z \in x_0v_2$ we can replace x by x_0. (Remark that if $x_0v \cap yz \neq \emptyset$ then the intersection $xv \cap yz$ is non-empty too.) So (xyz) is a metric triangle of the space X. By Lemma 6 iii) $d(y, z) \leq d(v_1, v_2) = 2$ and thus the size of the metric triangle (xyz) is 1 or 2. In the first case from Lemma 10 we infer that the set x/yz is convex.

Next assume that (xyz) is a metric triangle of size 2. By Lemma 7 ii) we have $d(x, u) = 2$ for each point $u \in yz$. By the triangle condition there exist a point y_0 adjacent to x, y, u and a point z_0 adjacent to x, z, u (Fig.11). We claim that $d(y_0, v_2) = d(x, v_2)$ and $d(z_0, v_1) = d(x, v_1)$. Assume the contrary and let $d(y_0, v_2) < d(x, v_2)$. Then $y_0, z_0 \in xv_2$. By the quadrangle condition there is a point $w \in y_0v_2 \cap z_0v_2$, adjacent to y_0 and z_0. If $w = u$ then $u \in yz \cap xv_2$, yielding a contradiction to the assumption that $yz \cap xv_2 = \{z\}$. Thus $d(y_0, v_2) = d(x, v_2), d(z_0, v_1) = d(x, v_1)$, i.e. $u \in y_0v_2 \cap z_0v_1$ and so $v_1, v_2 \in (y_0/yu) \cap (z_0/zu)$. By Lemma 9 the sets y_0/yu and z_0/zu are convex, that is $y_0v \cap yu \neq \emptyset$ and $z_0v \cap zu \neq \emptyset$.

Note that either the points y_0 and z_0 are adjacent or by the triangle condition we can find a new point z_1 adjacent to z, x and y_0. In the either case $v_1 \in x/y_0, v_2 \in x/z_0$. By Lemma 10 we conclude that $v \in x/y_0 \cup x/z_0$. Assume, without loss of generality, that $v \in x/y_0$, i.e. $y_0 \in vx$. Since $y_0v \cap z_1y_0 \neq \emptyset$ and $u \in yz$ we deduce that $xv \cap yz \neq \emptyset$. In the second case we have $v_1, v_2 \in x/z_1y_0$. By Lemma 10 we conclude that $v \in x/z_1y_0$, i.e. $vx \cap z_1y_0 \neq \emptyset$. If $y_0 \in vx$ then $vx \cap zy \supset y_0v \cap yu \neq \emptyset$. Now let $z_1 \in vx$. Note that $d(z_1, v_1) = d(x, v_1)$ (the proof of this fact is similar). Hence $u \in z_1v_1$ and therefore by Lemma 9 we obtain $z_1v \cap zu \neq \emptyset$. So $vx \cap yz \supset vz_1 \cap zu \neq \emptyset$, thus completing the proof of the theorem □

ACKNOWLEDGEMENT I am indebted to the referee who encovered a number of inaccuracies and corrected linguistic errors in the original version of the paper.

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Eingegangen am 10. Dezember 1990; in revidierter Form am 27. Oktober 1993