On Covering Bridged Plane Triangulations with Balls

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Abstract: We show that every plane graph of diameter $2r$ in which all inner faces are triangles and all inner vertices have degree larger than 5 can be covered with two balls of radius r. © 2003 Wiley Periodicals, Inc. J Graph Theory 44: 65–80, 2003

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1. INTRODUCTION

In this note, we prove the following result:

Theorem 1.1. *The vertex-set of every plane graph* $G = (V, E)$ *of diameter* $2r$ *in which all inner faces are triangles and all inner vertices have degrees larger than 5 can be covered with two balls of radius* r.

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This provides a partial answer to a question of C. Gavoille [9] (cf. also Conjecture 2.11 of [10]) whether there is a constant c such that every planar graph of even diameter $D := 2r$ can be covered with at most c balls of radius r. The problem of covering planar graphs of diameter 2 and 3 with a fixed number of balls of radius 1 has been considered in [13]. Recently, Gavoille et al. [10] proved that for every planar graph G of diameter D, for every $\varepsilon > 0$, and for every integer $r \geq (5/7 + \varepsilon)D$, G can be covered with at most $3/\varepsilon + 6$ balls of radius r. They also motivate the practical interest of this question by establishing a close relationship between the conjecture above and the existence of routing tables of small dilation and fixed compactness.

Plane triangulations in which all inner vertices have degrees ≥ 6 represent a natural class of planar graphs from the metric and the geometric points of view (to give an instructive example, consider the regular triangular grid and any of its subgraphs induced by the vertices lying on a simple circuit and inside the region bounded by this circuit). Namely, from a result of [2] follows that they are precisely the planar $K_{1,1,3}$- and K_4-free bridged graphs, one of the basic classes of graphs in metric graph theory (recall that bridged graphs are those graphs in which all isometric cycles have length 3 [11,14]). Besides nice metric and convexity properties, bridged graphs obey some interesting collapsibility properties related with the game of a cop and a robber [1,6]. On the other hand, they give rise to important simplicial complexes [7] (see also [8] for the 2-dimensional case). For example, simplicial complexes obtained from K_4-free bridged graphs are precisely the 2-dimensional simplicial complexes which can be endowed with an intrinsic metric of non-positive curvature [7]. As such, this class of triangulations has been investigated in [4,8,12]. The proof of our theorem is based on the fact that the plane triangulations in question are bridged and uses several metric and convexity properties of these graphs established in [2–5, 11,12,14].

Gavoille et al. [10] constructed a plane graph of diameter 8 (called the "globe graph") for which the minimum number of balls of radius 4 covering the graph is 4, thus establishing that $c \geq 4$. Nevertheless, we believe that $c \leq 3$ holds for large classes of plane graphs and that our approach can be adapted to such instances, in particular to plane graphs of type (p, q) investigated in [4]. Let p, q be natural numbers which satisfy $1/p + 1/q = 1/2$. The plane graph G is *of type* (p, q) [4] if each vertex is incident to at least p edges and every interior face is bounded by at least q edges. Bridged plane triangulations considered in our note are exactly the plane triangulations of type $(6, 3)$. Using our approach one can show that $c \leq 2$ holds for all plane quadrangulations of type $(4, 4)$ (the proof is much easier than for triangulations).

The rest of the paper is organized as follows. In Section 3, we present the proof of the theorem. In Section 2, we collect several facts about plane bridged triangulations employed in this proof. To make the paper self-contained, we provide the proof for most of these facts in an Appendix.

2. AUXILIARY FACTS

Let $G=(V,E)$ be a connected finite graph endowed with the standard graph-distance $d=d_G$. The (metric) *interval* $I(u,v)$ between two vertices u and v consists of all vertices on shortest (u,v)-paths, that is:

$$I(u,v)=\{x\colon d(u,x)+d(x,v)=d(u,v)\}.$$

For convenience we will use the short-hand $[u,v]$ to denote an interval consisting of a single shortest (u,v)-path. A subset of vertices A (or the subgraph induced by A) of G is *convex* if the interval $I(u,v)$ between any two vertices u and v of A lies entirely in A. The *convex hull* of a subset S is the smallest convex set of G containing S. For a set $S\subset V$ and a vertex $v\notin S$ we denote by

$$\pi(v,S)=\{x\in S\colon d(v,x)=d(v,S)\}$$

the (metric) *projection* of v on S (recall that $d(v,S)=\min\{d(v,y):y\in S\}$). An induced subgraph H of G is an *isometric* subgraph if $d_G(u,v)=d_H(u,v)$ for every two vertices u,v in H. A graph is called *bridged* if it does not contain any isometric cycle of length greater than 3. A *ball* of radius k centered at the vertex v is the set $B_k(v)=\{x\in V\colon d(v,x)\leq k\}$ (set $N(v):=B_1(v)\setminus\{v\}$). More generally, for a set $S\subseteq V$ and an integer k let $B_k(S)=\{x\in V:d(x,S)\leq k\}$. We write $u\sim v$ if the vertices u and v are adjacent and $u\nsim v$ otherwise.

Fact 1 [2]. *Every plane graph in which all interior faces are triangles and all inner vertices (i.e., the vertices not incident with the outer face) have degrees larger than 5 is a K_4-free bridged graph.*

In view of this result, we will call the plane graphs obeying this condition *plane bridged triangulations.*

Notice that in a plane bridged triangulation G every 3-cycle bounds an interior face unless the graph itself is a 3-cycle. Indeed, pick a 3-cycle C and consider the subgraph G' induced by all vertices lying on C or inside the region bounded by C. By Fact 1, G' is a bridged triangulation whose outer face is C. Let n_2 and n_3 be the numbers of vertices of G' with degrees 2 and 3. Clearly $n_2+n_3\leq 3$. From Lemma 7 of [2], we infer that $2n_2+n_3\geq 6$, whence $n_2=3$ and $n_3=0$, and therefore $G'=C$.

The following characterization of bridged graphs is a basic tool in our proof.

Fact 2 [11,14]. *A graph G is bridged if and only if for every convex set S of G and every integer k the set $B_k(S)$ is convex. In particular, all balls $B_k(x)$ of bridged graphs are convex.*

Fact 2 implies that the neighbors of x in the interval $I(x,y)$ are pairwise adjacent. We continue with the following consequence of Fact 2:

Fact 3 [5]. *For any three vertices u,v,w of a bridged graph G with $1=d(v,w)<d(u,v)=d(u,w)=k$ there exists a common neighbor x of v and w such that $d(u,x)=k-1$.*

Fact 4 [5]. *If S is a convex set of a bridged graph G and $u \notin S$, $v \in S$, then there exists a vertex $x \in \pi(u, S)$ such that $d(u, v) = d(u, x) + d(x, v)$.*

For the remainder of this note, G will be a plane bridged triangulation and ∂G will be the outer face of G.

Three vertices u, v, w of a graph G are said to form a *metric triangle uvw* if the intervals $I(u, v), I(v, w)$, and $I(w, u)$ pairwise intersect only in the common end vertices, i.e., $I(u, v) \cap I(u, w) = \{u\}, I(v, u) \cap I(v, w) = \{v\}$, and $I(w, u) \cap I(w, v) = \{w\}$. If $d(u, v) = d(v, w) = d(w, u) = k$, then this metric triangle is called *equilateral* of *size k*. A metric triangle uvw is a *quasi-median* of the triplet x, y, z if

$$\begin{aligned} d(x, y) &= d(x, u) + d(u, v) + d(v, y), \\ d(y, z) &= d(y, v) + d(v, w) + d(w, z), \\ d(z, x) &= d(z, w) + d(w, u) + d(u, x). \end{aligned}$$

Any triplet x, y, z of vertices of an arbitrary connected graph G admits at least one quasi-median. Indeed, let u be a vertex in $I(x, y) \cap I(x, z)$ that is furthest from x, let v be a vertex in $I(y, u) \cap I(y, z)$ that is furthest from y, and let w be the vertex from $I(z, u) \cap I(z, v)$ that is furthest from z. Then one can easily check that uvw is a metric triangle and that all three distance equalities are satisfied. (It is shown in [3] that every triplet of vertices of a plane bridged triangulation has a unique quasi-median, however this property is not used in our proof.)

Fact 5 [3]. *If uvw is a metric triangle of G, then it is equilateral of some size k and the convex hull $T(u, v, w)$ of the vertices u, v, w is isomorphic to the graph of the tiling of the equilateral (plane) triangle of size k into equilateral triangles of size 1.*

Fact 6 [4,12]. *For each vertex v of G, all vertices at maximum distance from v are located on the outer face ∂G.*

Fact 7. *Any interval $I(x, y)$ of G is convex. Additionally, there exist two non-crossing shortest paths P' and P'' between x and y such that $I(x, y)$ consists of all vertices of G located in the region $\mathcal{R}(x, y)$ of the plane bounded by the closed walk $P' \cup P''$.*

Fact 8. *For every pair of vertices x, y and any $0 \leq i \leq d(x, y)$ the level set $L_i := \{z \in I(x, y) : d(x, z) = i\}$ of $I(x, y)$ induces a convex path with the end-vertices $u_i \in P'$ and $v_i \in P''$.*

Call the subgraph S_i of G induced by two consecutive levels L_i and L_{i+1} of $I(x, y)$ a *strip*. Obviously, each S_i is convex and contains exactly two corners (vertices of degree 2): S_i is maximal outer-planar (so it contains at least two corners) and the levels are convex (so it contains at most two corners). If $|L_i| = |L_{i+1}|$, then this pair of corners is either $\{u_i, v_{i+1}\}$ or $\{v_i, u_{i+1}\}$. Otherwise, if $|L_i| < |L_{i+1}|$, then u_{i+1} and v_{i+1} are the only corners of S_i (analogously, if $|L_{i+1}| < |L_i|$, then u_i and v_i are the only corners of S_i).

For a vertex $p \notin I(x,y)$, let $p'x(p)y(p)$ be a quasi-median of the triplet p,x,y.

Fact 9. *If both $x(p)$ and $y(p)$ belong to the sub-path of P' between u_i and y, then $x(p) \in I(p,u_i) \cap I(p,v_i)$.*

3. PROOF OF THE THEOREM

Let G be a plane bridged triangulation of diameter $d(G) = 2r$. We start with a special case for which one can explicitly locate the centers of the two r-balls covering G.

Lemma 3.1. *If G contains three vertices x,y,z such that $d(x,y) = d(y,z) = d(z,x) = 2r$, then G can be covered with two r-balls.*

Proof. Let $x'y'z'$ be a quasi-median of the triplet x,y,z. Since the metric triangle $x'y'z'$ is equilateral by Fact 5, we deduce that $d(x,x') = d(y,y') = d(z,z')$. This implies that the size of $x'y'z'$ is an even number, say $2k$. Let u, v, and w be the middle-vertices of the convex paths $[x',y'], [y',z']$, and $[z',x']$, respectively; see Figure 1. We assert that any two of the r-balls centered at u, v, and w cover the graph G. Fact 5 implies that any two such balls cover $T(x',y',z')$: each of them covers $T(u,v,w)$, while $T(x',u,w) \subseteq B_r(u) \cap B_r(w), T(y',u,v) \subseteq B_r(u) \cap B_r(v)$, and $T(z',v,w) \subseteq B_r(v) \cap B_r(w)$. Now, pick a vertex $p \notin T(x',y',z')$.

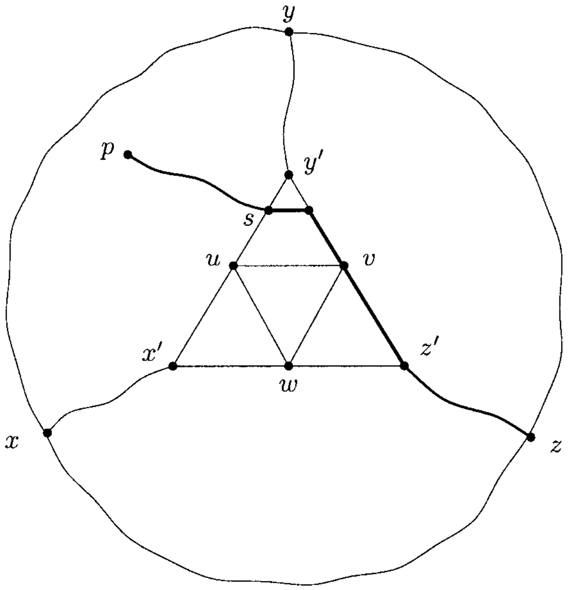

FIGURE 1.

Without loss of generality assume that p belongs to the region bounded by the (x,y)-sub-path of ∂G not passing through z and a shortest path P between x and y traversing the vertices x' and y'. Any shortest path between z and p intersects P in some vertex s. Suppose without loss of generality that s belongs to the sub-path of P between u and y. From Facts 4 and 5, we infer that s, v, and z' lie on a common shortest path between p and z. Since $d(p,z) \leq 2r$ and $d(v,z) = r$, necessarily $d(v,p) \leq r$. If s belongs to the sub-path of P comprised between y and y', then $d(u,s) = d(v,s)$, otherwise, if $s \in I(u,y')$, then $d(u,s) \leq d(u,y') = k = d(v,s)$. In both cases we deduce that $d(u,p) \leq d(u,s) + d(s,p) \leq d(v,p) \leq r$, thus establishing our assertion. ■

Now, we consider the general case: for a diametral pair x,y of G and any vertex $z \in V$ we have $\min\{d(z,x), d(z,y)\} < 2r$. Fact 6 yields $x,y \in \partial G$. Select the shortest (x,y)-paths P' and P'' as in Fact 7. Let $u*$ and $v*$ be the middle-vertices of P' and P'', respectively (i.e., $u*$ and $v*$ are the end-vertices of the level $L_r = [u*, v*]$) . Notice that the r-ball centered at any vertex $w \in L_r$ covers the whole interval $I(x,y)$: the ball $B_r(w)$ is convex and $x,y \in B_r(w)$, whence $I(x,y) \subseteq B_r(w)$.

The definition of the level set L_i implies that the quasi-medians of the triplets x, u_i, v_i and y, u_i, v_i have the form $x'u_iv_i$ and $y'u_iv_i$. From Fact 5 we conclude that $d(u_i, v_i) \leq \min\{i, 2r - i\}$. Since x,y is a diametral pair, from Fact 6 we deduce that either $x(p), y(p) \in P'$ or $x(p), y(p) \in P''$. Denote by V' and V'' the sets of vertices p of the first and of the second type, respectively. Let V'_0 be the set of all vertices $p \in V'$ such that the vertex $u*$ is an interior vertex of the convex path $[x(p), y(p)]$.

We have all prerequisites to accomplish the proof of the Theorem. We will show that the centers of two r-balls covering G can be located either at the vertices $u*$ and $v*$, or one center will be on the convex path $L_r = [u*, v*]$ and another one outside the interval $I(x,y)$. If $d(u*,p) \leq r$ and $d(v*,q) \leq r$ for any vertices $p \in V'$ and $q \in V''$, then $u*$, $v*$ constitutes the desired pair of centers. So, assume that there is a vertex $p \in V'$ with $d(u*,p) > r$. Since $d(x,p) \leq 2r$ and $d(y,p) \leq 2r$, necessarily $u*$ belongs to the convex path $[x(p), y(p)]$, moreover $u* \neq x(p), y(p)$. Fact 5 implies that $d(p, I(x,y)) = d(p, u*) > r$. Now, if there exists a vertex $q \in V''$ with $d(q, v*) > r$, we can show in a similar way that $d(q, I(x,y)) = d(q, v*) > r$. Since any shortest path between p and q necessarily intersects the paths P' and P'', we infer that $d(p,q) > 2r$, a contradiction. Hence $d(v*, q) \leq r$ for any $q \in V''$.

We describe now how to find the center located outside $I(x,y)$. Among the vertices of V'_0, choose a vertex s such that $\min\{d(u*, x(s)), d(u*, y(s))\}$ is as large as possible. Suppose without loss of generality that $d(u*, y(s)) \leq d(u*, x(s))$. Then locate the center at the vertex $s* \in [s', y(s)]$ at distance $d(u*, y(s))$ from $y(s)$.

Lemma 3.2. *The ball $B_r(s*)$ covers the set V'_0.*

Proof. Pick an arbitrary vertex $p \in V'_0$. If $p \in T(s', x(s), y(s))$, then $p \in I(s', u^*)$. Since $s', u^* \in B_r(s^*)$ and the ball $B_r(s^*)$ is convex, necessarily $p \in B_r(s^*)$. So, assume that $p \notin T(s', x(s), y(s))$. Then either $x(p) \in [x(s), u^*]$ or $y(p) \in [u^*, y(s)]$. Let $t := [s', y(s)] \cap [x(p), p']$ in the first case and $t := [s', x(s)] \cap [p', y(p)]$ in the second case; both possibilities are illustrated in Figures 2 and 3.

First, suppose that $x(p) \in [x(s), u^*]$. Let p^* be the vertex of $[x(p), p']$ at distance $d(x(p), u^*)$ from $x(p)$. Since $d(x, p^*) = d(x, u^*) = r$ and $p^* \in I(p, x)$, we obtain $d(p^*, p) \leq r$. On the other hand, from the choice of s^* we infer that $d(p^*, x(p)) \leq d(s^*, y(s))$. Since $p^* \in [t, x(p)]$, $s^* \in [t, y(s)]$ and $d(t, x(p)) = d(t, y(s))$, we deduce that $d(s^*, t) \leq d(p^*, t)$. As $t \in I(p^*, p)$ and $d(p^*, p) \leq r$, necessarily $d(s^*, p) \leq r$.

Now, assume that $y(p) \in [u^*, y(s)]$. Let p^* be the vertex of $[y(p), p']$ at distance $d(y(p), u^*)$ from $y(p)$. Clearly $p^* = [u^*, s^*] \cap [p', y(p)]$ and $d(p^*, p) \leq r$. Pick the vertex $w \in [p^*, t]$ at distance $d(p^*, s^*)$ from p^* and s^* (w is well-defined due to the form of $T(s', x(s), y(s))$ established in Fact 5). Since w and t lie on a common shortest path between p^* and p, we infer that $d(s^*, p) \leq d(p^*, p) \leq r$. ■

Thus $V'_0 \subseteq B_r(s^*)$, and therefore the balls $B_r(s^*), B_r(u^*)$, and $B_r(v^*)$ cover the whole graph G. We will show that instead of $B_r(u^*)$ and $B_r(v^*)$ there is an r-ball centered at some vertex of $[u^*, v^*]$ which solely covers the set $V - V'_0$. For the rest of this proof, fix a vertex z of V'_0 such that $d(z, u^*) > r$. For a vertex $t \in V - V'_0$, set $J_t := B_r(t) \cap [u^*, v^*]$. Since $d(t, u^*) \leq r$ if $t \in V'$ and $d(t, v^*) \leq r$ if $t \in V''$, every J_t is a sub-path of $[u^*, v^*]$ containing at least one of the vertices

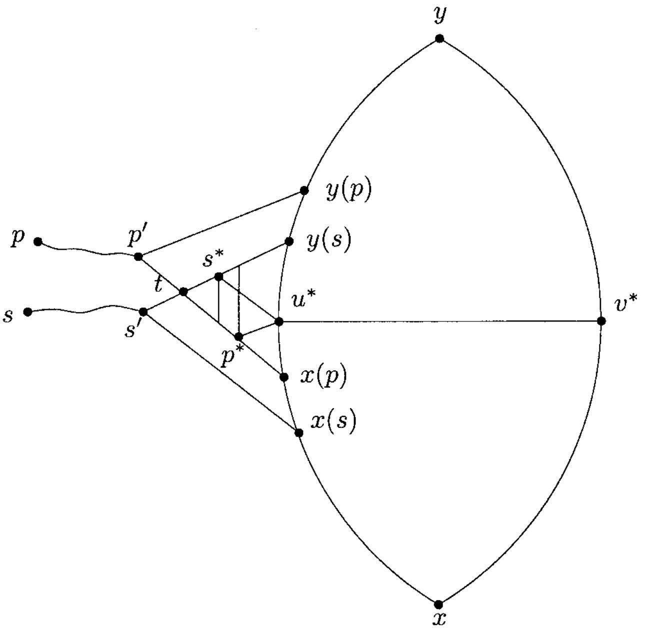

FIGURE 2. $x(p) \in [x(s), u^*]$.

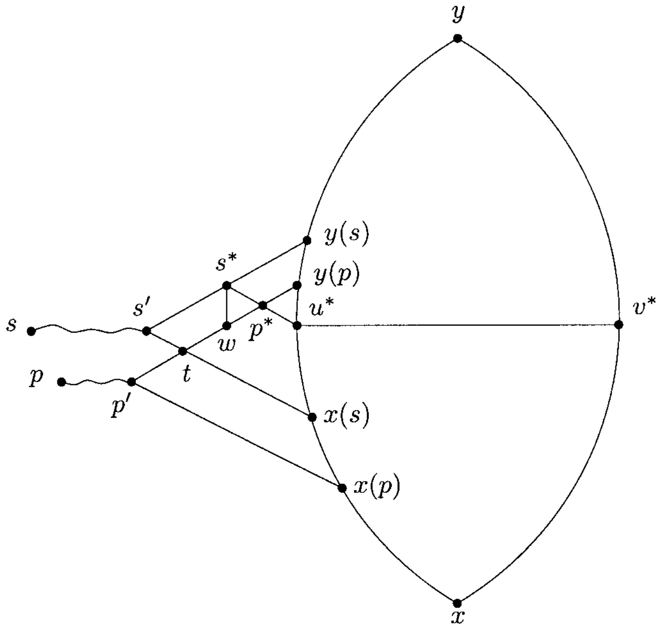

FIGURE 3. $y(p) \in [u^*, y(s)]$.

u^*, v^*. Since $I(x,y) \subseteq B_r(w)$ for every $w \in [u^*, v^*]$, necessarily $J_t = [u^*, v^*]$ for any $t \in I(x,y)$. It suffices to verify that the paths $\{J_t : t \in V - V_0'\}$ pairwise intersect, because any pairwise intersecting sub-paths of a path have a vertex in common (the Helly property).

Lemma 3.3. *The paths of the collection* $\{J_t : t \in V - V_0'\}$ *pairwise intersect.*

Proof. Suppose by way of contradiction that there exist two vertices $p, q \in V - V_0'$ such that $J_p \cap J_q = \emptyset$. We may assume that $p \in V'$ and $q \in V''$, because $u^* \in J_p \cap J_q$ if $p, q \in V'$ and $v^* \in J_p \cap J_q$ if $p, q \in V''$. Additionally, let us suppose without loss of generality that the vertices $x(p)$ and $y(p)$ belong to the sub-path of P' between u^* and y. Let $J_p := [u^*, v]$ and $J_q := [u, v^*]$.

Let $p_0u^*v^+$ be a quasi-median of the triplet $x(p), u^*, v^*$. Fact 9 implies that $p_0u^*v^+$ is also a quasi-median of the triplet p, u^*, v^*. Since $d(p, v^+) = d(p, u^*) \leq r$, we deduce that $v^+ \in [u^*, v]$. Since the sets $V' \cup I(x,y)$ and $V'' \cup I(x,y)$ are convex and $\pi(z, V'' \cup I(x,y)) = [x(z), y(z)]$, $\pi(q, V' \cup I(x,y)) = [x(q), y(q)]$, from Fact 4 we infer that there exists a shortest (q,z)-path which passes through q' and intersects the convex paths $[x(q), y(q)]$ and $[x(z), y(z)]$ in vertices v_i and u_j, respectively. Analogously, we will find a shortest (p,q)-path which passes through p' and q' and intersects the convex paths $[x(p), y(p)]$ and $[x(q), y(q)]$ in vertices u_l and v_k, respectively. We distinguish three cases depending on the position of vertex q; see Figures 4–7 for an illustration.

Case 1. *$x(q)$ belongs to the sub-path of P'' between v^* and y.*

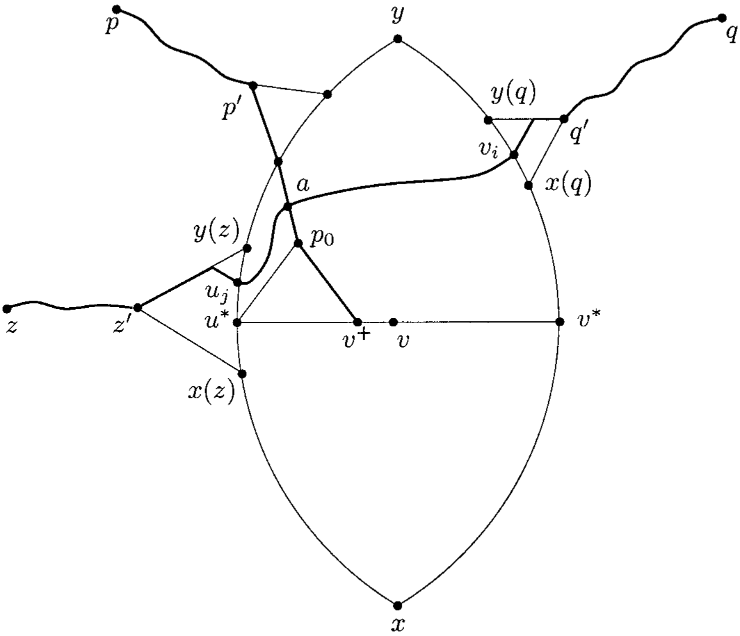

FIGURE 4. Case 1.

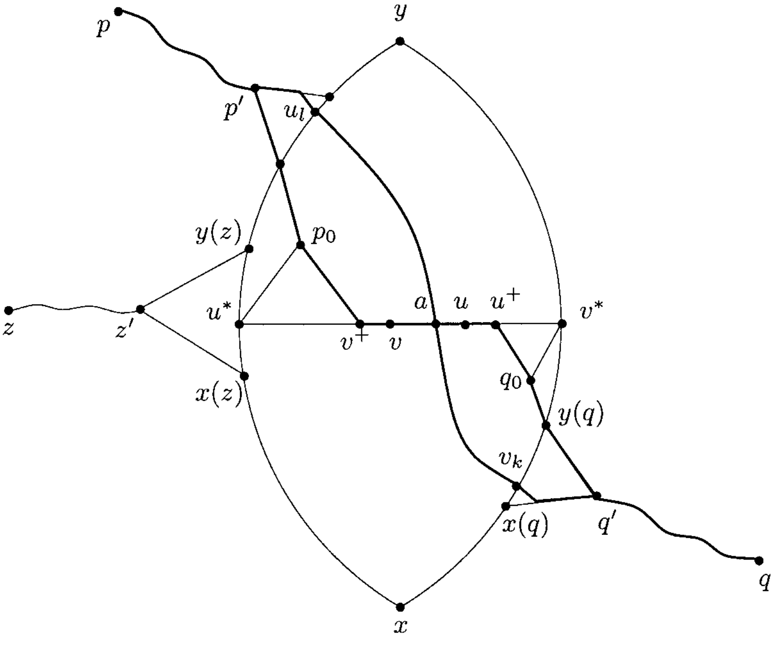

FIGURE 5. Case 2.

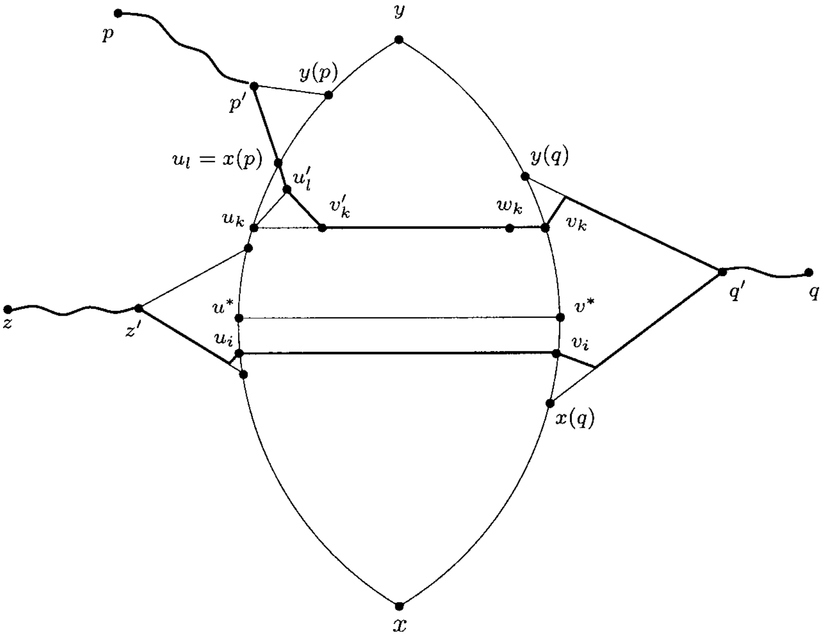

FIGURE 6. Case 3.

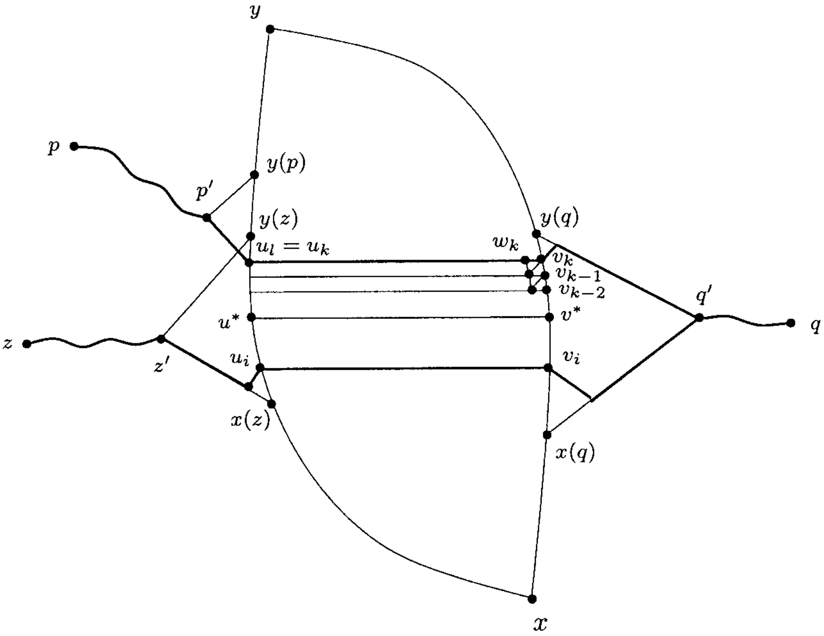

FIGURE 7. Case 3.

Since $d(z, I(x, y)) > r$, necessarily $d(q, u_j) < r$. Any shortest (v_i, u_j)-path intersects a shortest (p, v^+)-path passing via $p', x(p)$, and p_0 in some vertex a. Obviously $d(q, a) \leq d(q, u_j)$, therefore $a, v^* \in B_r(q)$. Since $v^+ \in I(a, v^*)$ and $B_r(q)$ is convex, we infer that $v^+ \in B_r(q)$. Thus $v^+ \in J_p \cap J_q$, a contradiction.

Case 2. *there exists a shortest* (u_l, v_k)*-path intersecting* $[u^*, v^*]$ *in a vertex a.*

Then either $y(q)$ belongs to the sub-path of P'' comprised between v^* and x or v^* belongs to the convex path $[x(q), y(q)]$. Let $q_0u^+v^*$ be a quasi-median of the triplet q, u^*, v^*. In the either case, Fact 9 implies that one can take q_0u^+v as a quasi-median of the triplet $y(q), u^*, v^*$. In analogy with v^+, one can show that in this case $u^+ \in [u, v^*]$. On the other hand, if $v^* \in [x(q), y(q)]$, then $v^* \in I(q, u^*)$ because $d(u^*, v^*) \leq d(u^*, v_m)$ and $d(q, v^*) = d(q, v_m)$ hold for any vertex $v_m \in [x(q), y(q)]$. This yields $u^+ = v^*$ in this case.

Now, consider the vertex a. If $a \in [v^+, u^+]$, then $d(p, a) = d(p, v^+) + d(v^+, a)$ and $d(q, a) = d(q, u^+) + d(u^+, a)$. Since $d(v^+, a) + d(a, u^+) = d(v^+, v) + d(v, u) + d(u, u^+)$ and $u \neq v$, we get

$$\begin{aligned} d(p, q) &= d(p, v^+) + d(v^+, v) + d(v, u) + d(u, u^+) \\ &\quad + d(u^+, q) = r + d(v, u) + r > 2r, \end{aligned}$$

a contradiction. Otherwise, if $a \in [u^*, v^+]$ (the case $a \in [v^*, u^+]$ is analogous), then $d(p, a) = d(p, v^+)$ and $d(a, q) \geq d(v^+, q)$. Hence $a = v^+$, and we are back in the preceding subcase.

Case 3. v^* *is an interior vertex of the path* $[x(q), y(q)]$.

By Fact 4, there is a shortest (q, z)-path intersecting the projections $\pi(z, I(x, y)) = [x(z), y(z)]$ and $\pi(q, I(x, y)) = [x(q), y(q)]$ in two vertices u_j and v_i, respectively. Notice that one can choose u_j and v_i so that $i = j$. Indeed, this is a consequence of the inequalities $d(u_i, v_i) \leq d(u_i, v_j)$ and $d(u_i, v_i) \leq d(u_j, v_i)$ which hold for all i and j. Additionally, assume that among all such pairs, the couple u_i, v_i is selected so that to minimize $|i - r|$. In other words, among all levels of $I(x, y)$ which are sub-paths of shortest (z, q)-paths, we select a level L_i located as close as possible to $L_r = [u^*, v^*]$. If $|L_r| = |L_i|$, then u^*, v^* lie on a shortest (q, z)-path, yielding $d(q, u^*) \leq r$, a contradiction. Hence $|L_i| < |L_r|$. If $i > r$, then the paths $[p', v^+]$ and L_i have a vertex a in common. As $a \in I(z, q)$ and $d(z, a) \geq d(z, u^*) > r$, we obtain that $d(q, a) < r$. Since $a, v^* \in B_r(q)$ and $v^+ \in I(a, v^*)$, necessarily $v^+ \in B_r(q)$, contrary to the choice of p and q. Thus $< r$. Now, take the vertices u_l and v_k on a shortest (p, q)-path L so that k is as small as possible. Obviously $k \leq l$. If $k \leq r$, then the sub-path of L comprised between u_l and v_k intersects the path $[u^*, v^*]$, therefore the conditions of Case 2 are fulfilled. Hence $r < k \leq l$.

Consider the levels $L_i, L_{i+1}, \ldots, L_r, L_{r+1}, \ldots, L_k$. We claim that

$$d(u_i, v_i) < d(u_{i+1}, v_{i+1}) \leq \cdots \leq d(u^*, v^*) \leq \cdots \leq d(u_k, v_k). \qquad (*)$$

The choice of u_i and v_i yields the first inequality. To verify the remaining inequalities, consider the path $[v_i, v_k]$ (which is convex as a sub-path of the convex path $[x(q), y(q)]$). Fact 4 and the convexity of levels (Fact 8) imply that $d(u_m, [v_i, v_k]) = d(u_m, v_m)$ for every $i \leq m \leq k$. If $d(u_i, v_i) < d(u_m, v_m) > d(u_{m+1}, v_{m+1})$ for some $i < m < k$, then $u_i, u_{m+1} \in B_\delta([v_i, v_{m+1}])$, where $\delta := \max\{d(u_i, v_i), d(u_{m+1}, v_{m+1})\}$. Since $u_m \in [u_i, u_{m+1}] - B_\delta([v_i, v_{m+1}])$, we obtain a contradiction with Fact 2. This establishes the required property $(*)$.

From $(*)$ one may conclude that $u_k \neq v_k$. Let w_k be the neighbor of v_k in the level L_k. We assert that $d(p, w_k) < d(p, v_k)$. If the vertex u_k belong to the sub-path of P' comprised between $x(p)$ and u^* (see Figure 6), then Fact 9 implies that $x(p) \in I(p, u_k) \cap I(p, v_k)$, therefore we may suppose that $u_l = x(p)$. In this case, a quasi-median of the triplet $u_l = x(p), u_k, v_k$ (and therefore of the triplet p, u_k, v_k) has the form $u'_l u_k v'_k$, where $v'_k \in [u_k, v_k]$. If $v'_k \neq v_k$, then v'_k and w_k lie on a common shortest path between p and v_k, and we are done. On the other hand, if $v'_k = v_k$, then $d(p, u_k) = d(p, v_k)$: indeed, the metric triangle $u'_l u_k v'_k$ is equilateral, thus

$$\begin{aligned} d(p, u_k) = d(p, x(p)) + d(x(p), u'_l) + d(u'_l, u_k) = d(p, x(p)) \\ + d(x(p), u'_l) + d(u'_l, v'_k) = d(p, v_k). \end{aligned}$$

In this case, since

$$r \geq d(p, u^*) = d(p, u_k) + d(u_k, u^*) = d(p, v_k) + d(v_k, v^*) \geq d(p, v^*),$$

we obtain that $v^* \in B_r(p) \cap B_r(q)$, contrary to our assumption. Now, assume that u_k belongs to $[x(p), y(p)]$ (see Figure 7). Then, similarly to the proof of the equality $i = j$, one can show that $k = l$. Hence u_k and w_k lie on a common shortest path between p and v_k, thus $d(p, w_k) < d(p, v_k)$.

Consider the strip S_{k-1}. If v_k is a corner of S_{k-1}, then v_{k-1} is adjacent to w_k. Since $d(p, w_k) < d(p, v_k)$ in both cases, this would imply that $d(p, v_{k-1}) \leq d(p, v_k)$, contrary to the choice of v_k. From the properties of strips noticed above and $(*)$, we obtain that $|L_{k-1}| = |L_k|$ and that v_{k-1} is a corner of S_{k-1}, hence v_k is adjacent to the neighbor w_{k-1} of v_{k-1} in L_{k-1}. Consequently, if v_{k-1} is a corner of S_{k-2}, then v_{k-2} and w_{k-1} are adjacent, contradicting the convexity of the path $[v_i, v_k]$. Thus v_{k-2} is a corner of S_{k-2}. The property $(*)$ implies that $|L_{k-2}| = |L_{k-1}|$. Continuing this way, we deduce that every v_m $(k > m \geq i)$ is a corner in the strip S_m and that the paths L_{m+1} and L_m have equal length. Therefore all levels $L_k, L_{k-1}, \ldots, L_r, \ldots, L_i$ have the same size, contrary to $(*)$. Thus the paths from $\{J_t : t \in V - V'_0\}$ pairwise intersect, completing the proof of the lemma. ■

By Helly property we will find a vertex $w^* \in \cap\{J_t : t \in V - V'_0\}$. Clearly $V - V'_0 \subseteq B_r(w^*)$. Since $V'_0 \subseteq B_r(s^*)$, this completes the proof of the Theorem. ■

APPENDIX

Proof of Fact 3. Pick $v' \in I(v,u) \cap N(v)$ and $w' \in I(w,u) \cap N(w)$. If $v' = w'$ we are done. Otherwise, since $v', w' \in B_{k-1}(u), v, w \notin B_{k-1}(u)$ and the ball $B_{k-1}(u)$ is convex, we conclude that $d(v',w') \leq 2$. If v' and w' are adjacent we obtain a 4-cycle (v', w', w, v), which cannot be induced. Hence either $w \sim v'$ or $v \sim w'$. In both cases, we get a common neighbor of v, w at distance $k-1$ to u. Finally, suppose that $d(v',w') = 2$ and let x be a common neighbor of v' and w'. Clearly $x \in I(v',w') \subset B_{k-1}(u)$. Consider the 5-cycle (v', x, w', w, v), which cannot be induced. Since $v \not\sim w'$ and $w \not\sim v'$ (otherwise $B_{k-1}(u)$ will be not convex), we conclude that $v \sim x \sim w$. ■

Proof of Fact 4. Suppose by way of contradiction that there exists a vertex $v \in S$ such that $I(u,v) \cap \pi(u,S) = \emptyset$. Among all such vertices of S assume that v is as close as possible to $\pi(u,S)$. Let x be a vertex of $\pi(u,S)$ such that $d(v,x) = d(v,\pi(u,S))$. Pick a neighbor v' of v in $I(v,x) \subseteq S$. The choice of v yields $I(v',u) \cap \pi(u,S) \neq \emptyset$. Obviously $x \in I(v',u)$. From the choice of v we infer that $d(v',u) \geq d(v,u)$, hence $v' \neq x$. Let v'' be a neighbor of v' in $I(v',x) \subset I(v',u)$. If $d(v',u) > d(v,u)$, then $v \sim v''$ because $v, v'' \in I(v',u)$, contrary to the choice of v'. Hence $d(u,v) = d(u,v')$. By Fact 3, there is a common neighbor y of v and v' one step closer to u. Since $y, v'' \in I(v',u)$, $y \sim v''$. Hence $y \in I(v,v'') \subseteq S$. Since $y \in I(v,x)$, the choice of v yields $x \in I(y,u)$. But this is impossible, because $d(y,u) < d(v',u)$ and $d(y,x) = d(v',x)$. ■

Proof of Fact 5. Assume that $k := d(u,v) \geq d(u,w)$. First we show that all vertices of $I(v,w)$ have distance k to u. Since the ball $B_k(u)$ is convex and $v, w \in B_k(u)$, all vertices of $I(v,w)$ have distance $\leq k$ to u. Suppose by way of contradiction that there exists a vertex $x \in I(v,w) \cap B_{k-1}(u)$, and assume that all vertices $y \in I(v,w)$ obeying $d(y,v) < d(x,v)$ have distance k to u. Pick a neighbor x' of x in $I(x,v)$. Then $x \in I(x',u)$ and $x' \neq v$ (otherwise $x \in I(v,u) \cap I(v,w)$). Let x'' be a neighbor of x' in $I(x',v)$. By Fact 3, x'' and x' have a common neighbor y at distance $k-1$ to u. Then $y, x \in I(x',u)$, therefore $y \sim x$. Since $y \sim x''$, we conclude that $y \in I(x'',x) \subset I(v,w)$ because x'' and x lie on a common shortest (v,w)-path. However this contradicts the choice of x. Thus all vertices of $I(v,w)$ have distance k to u, in particular the metric triangle uvw is equilateral.

Next we establish that $I(v,w)$ is a path. Suppose not: then one can easily see that $I(v,w)$ must contain two adjacent vertices z', z'' having the same distance to v (and to w). Previous assertion implies that z' and z'' have the same distance to u, too. Applying Fact 3, three times, we will find common neighbors v', w', u' of z', z'' one step closer to v, w, u, respectively. One can easily see that these vertices are pairwise distinct. From [2] we know that G does not contain induced $K_{1,1,3}$ (three triangles glued along a common edge). Hence at least two of the vertices u', v', w' are adjacent. But then we get a K_4, which is forbidden as well. This contradiction shows that the intervals $I(u,v), I(v,w), I(w,u)$ are convex paths.

To complete the proof, we proceed by induction on the size k of uvw. Let $[v, w] = (x_0 = v, x_1, \ldots, x_{k-1}, x_k = w)$. Applying Fact 3 to u and each pair x_{i-1}, x_i, we will find the vertices $y_1, \ldots, y_k$ at distance $k - 1$ to u, such that y_i is adjacent to x_{i-1} and x_i for each $i = 1, \ldots, k$. As $y_i, y_{i+1} \in I(x_i, u)$, the vertices y_i and y_{i+1} are adjacent (they cannot coincide because $I(v, w)$ is a path). Hence $d(y_1, y_k) = k - 1$. Since $y_1 \notin I(v, w)$, we obtain $d(y_1, w) = k$ and $x_1, y_2 \in I(y_1, w)$. Now, if $I(y_1, u) \cap I(y_1, y_k) \neq \{y_1\}$ and z is the neighbor of y_1 from this intersection, then $z \in I(y_1, y_k) \subset I(y_1, w)$. Hence $d(z, w) < k$, contrary to the fact that all vertices of $I(u, v)$ have distance k to w. Thus $I(y_1, u) \cap I(y_1, y_k) = \{y_1\}$. Analogously one can show that $I(y_k, u) \cap I(y_k, y_1) \neq \{y_k\}$, therefore the vertices y_1, y_k, u constitute a metric triangle of size $k - 1$. By the induction assumption, $T(y_1, y_k, u)$ is a triangle of size $k - 1$ of the regular triangular grid. Together with the path $[v, w]$ they induce in G a triangle T of size k of the same triangular grid. In order to prove the convexity of T in G, we apply Lemma 1 of [2] (which essentially coincides with Theorem 7 of [5]). According to this lemma, it suffices to show that $I(x, y) \subset T$ for any $x, y \in T$ at distance 2 in T. We already know that $T(y_1, y_k, u)$ and $[v, w]$ are convex, therefore let $x \in T(y_1, y_k, u)$ and $y \in [v, w]$. Since $d(x, u) \leq k - 1, d(y, u) = k$, and $d(x, y) = 2$, we conclude that x has distance $k - 1$ or $k - 2$ to u. Let z' be a common neighbor of x and y in T and suppose by way of contradiction that x and y have a common neighbor z outside T. This immediately implies that x and y cannot lie on a common side of T. For any other location of x and y in T, one can easily see that either x and y have a second common neighbor z'' in T or that z' is an inner vertex of degree 6 of T and G. In both cases the vertices z and z' are adjacent because $z, z' \in I(x, y)$. In the second case this leads to a contradiction because all neighbors of z' already belong to T. In the either case the vertices z, z', z'' are pairwise adjacent and together with one of the vertices x, y induce a forbidden K_4. This contradiction shows that T is convex. Since $u, v, w \in T \subseteq T(u, v, w)$, we deduce that $T(u, v, w) = T$. ■

Proof of Fact 7. To establish the convexity of $I(x, y)$ we again use Lemma 1 of [2]: pick $u, v \in I(x, y)$ at distance 2 and let w be their common neighbor. If $d(x, u) = d(x, v) =: k_1$ and $d(y, u) = d(y, v) =: k_2$, then $w \in B_{k_1}(x) \cap B_{k_2}(y)$ by convexity of balls, hence $d(x, w) + d(w, y) \leq k_1 + k_2 = d(x, y)$, i.e., $w \in I(x, y)$. Now assume that $d(x, u) = d(x, v) + 1$ and $d(y, v) = d(y, u) + 1$ (the remaining cases are trivial). Then one can assume that $d(x, w) = d(x, u)$ and $d(y, w) = d(y, v)$, otherwise we immediately conclude that $w \in I(x, y)$. By Fact 3, there is a vertex $x' \sim u, w$ one step closer to x and a vertex $y' \sim v, w$ one step closer to y. Since $v, x' \in I(w, x)$ and $u, y' \in I(w, y)$, we conclude that x' is adjacent to v and y' is adjacent to u. Since every 3-cycle bounds an inner face of G, we conclude that w is an inner vertex of degree 4, a contradiction. This establishes that $I(x, y)$ is convex.

Now we will show that if H is a plane bridged triangulation such that ∂H consists of two shortest paths P', P'' between two vertices $x, y \in \partial H$, then

$I(x,y) = H$. We proceed by induction on the number of vertices (or triangles) of the graph H, departing from the trivial case $P' = P''$. Assume that H is two-connected, otherwise we can apply the induction hypothesis to each 2-connected component of H. By Lemma 7 of [2], the numbers n_2 and n_3 of vertices of H with degrees 2 and 3 satisfy the inequality $2n_2 + n_3 \geq 6$. Since ∂H is constituted of two shortest (x,y)-paths, only x and y may have degree 2. Therefore there exists a vertex $z \in \partial H, z \neq x, y$, of degree 3, say $z \in P'$. Let u, v, w be the neighbors of z, where $u, w \in C$ and v is adjacent to u and w. Replacing in P' the vertex z by v, we will obtain a shortest path P'_0 between x and y. By the induction assumption, the vertices in the bridged triangulation bounded by the paths P'_0 and P'' belong to $I(x,y)$. Since $z \in I(x,y)$, we are done.

Let P' and P'' be two non-crossing shortest paths between x and y of G such that $I(x,y)$ contains a maximum number of vertices in the region $\mathcal{R}(x,y)$ bounded by the closed walk $P' \cup P''$. The sub-triangulation H of G contained in the region $\mathcal{R}(x,y)$ is bridged because any inner vertex of H is an inner vertex of G and therefore its degree is larger than 5. From previous assertion, we conclude that $H \subseteq I(x,y)$. It remains to show that $I(x,y) \subseteq H$. Suppose by way of contradiction that there exists a vertex $z \in I(x,y)$ outside $\mathcal{R}(x,y)$. Pick a shortest path P between x and y passing via z. Orient the paths P, P', and P'' from x to y. Let x' be the closest to z vertex of $(P' \cup P'') \cap P$ located before z, and let y' be the closest to z vertex of $(P' \cup P'') \cap P$ located after z. If x' and y' belong to the same path P' or P'', say $x', y' \in P'$, then replacing the sub-path of P' between x' and y' by the sub-path of P between these vertices, we will obtain a shortest (x,y)-path passing via z which together with P'' bounds a region containing $\mathcal{R}(x,y)$, a contradiction with the choice of P' and P''. Otherwise, if say $x' \in P' - P''$ and $y' \in P'' - P'$, let H' be the subgraph consisting of $P' \cup P''$ and the subpath of P from x' to y'. One of x or y, say x, is on the external face F of H' and the other, y, is an inner vertex of H'. But then y is further from x than any vertex of F, and so Fact 6 will not hold for x in the graph consisting of F and all vertices and edges of G inside F, a contradiction. This establishes the second assertion of Fact 7. ■

Proof of Fact 8. The set L_i is convex as the intersection of two convex sets $B_i(x)$ and $B_{d(x,y)-i}(y)$. Pick a shortest path Q_i between u_i and v_i. Since L_i is convex, Q_i intersects the paths P' and P'' only in its end-vertices. If there is a vertex $z \in L_i - Q_i$, then $z \in \mathcal{R}(x,y) - (P' \cup P'')$ and the path Q_i separates z from one of the vertices x and y, say the first. Then any shortest path between x and z will intersect Q_i, therefore $d(x,z) > i$, which is impossible. This shows that L_i is indeed a convex path. ■

Proof of Fact 9. By Fact 4, the interval $I(p,u_i)$ intersects the convex path $\pi(p, I(x,y)) = [x(p), y(p)]$. Since $x(p)$ is closer to u_i than any other vertex of $[x(p), y(p)]$, this intersection consists solely of $x(p)$. Analogously, by Fact 4 there is a shortest (p, v_i)-path intersecting $[x(p), y(p)]$ in some vertex u_j. This path will intersect the level L_m containing $x(p)$. Let s be the closest to $x(p)$ vertex from this intersection. From the choice of u_j and s and the convexity of the paths $[x(p), u_j]$ and $[x(p), s]$ one may conclude that $x(p)u_js$ is a metric triangle. From Fact 5 it follows that $d(s, x(p)) = d(s, u_j)$, yielding $x(p) \in I(p, s) \subseteq I(p, v_i)$. ■

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