Toroidal Embeddings of Right Groups

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Abstract

In this note we study embeddings of Cayley graphs of right groups on surfaces. We characterize those right groups which have a toroidal but no planar Cayley graph, such that the generating system of the right group has a minimal generating system of the group as a factor.

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1 Preliminaries

A graph is said to be *(2-cell-)embedded* in a surface M if it is "drawn" in M such that edges intersect only at their common vertices and deleting the graph from M yields a disjoint union of open disks. A graph is said to be *planar* if it can be embedded in the plane. By the *genus* of a graph X we mean the minimum genus among all surfaces in which X can be embedded. So if X is planar then the genus of X is zero. If a non-planar graph can be embedded on the torus, that is on the orientable surface of genus 1, it is called *toroidal*. A graph is said to be *outer planar* if it has an embedding in the plane such that one face is incident to every vertex.

It is known that each group can be defined in terms of generators and relations, and that corresponding to each such (non-unique) presentation there is a unique graph, called the Cayley graph of the presentation. A "drawing" of this graph gives a "picture" of the group from which certain properties of the group can be determined. The same principle can be used for other algebraic systems. So algebraic systems with a given system of generators will be called *planar* or *toroidal* if the respective Cayley graphs can be embedded on the plane or on the torus.

Finite planar groups have been cataloged by Maschke [6]. On the basis of Maschke's Theorem, in this work we investigate embeddings of certain completely regular semigroups (unions of groups), namely of right groups. This is a continuation of the investigations from [11] where Clifford semigroups were in focus. Here our attention is restricted to a special class of presentations of right groups for which we classify the toroidal right groups. Note that this generally only gives upper bounds on the genus of right groups. The full determination of the genus will be studied in a subsequent paper [4].

We use K_n for the complete graph on n vertices, C_n for the cycle on n vertices, and $K_{n,n}$ for the respective complete bipartite graph. We denote the cyclic group of order n by $\mathbb{Z}_n = \{0, \ldots, n-1\}$, and the dihedral, symmetric and alternating groups by D_n, S_n and A_n, respectively.

We recall that a *right group* is a semigroup of the form $G \times R_r$ where G is a group and R_r is a right zero semigroup, i. e., $R_r = \{r_1, \ldots, r_r\}$ with the multiplication $r_i r_j = r_j$ for $r_i, r_j \in R_r$.

Every semigroup presentation is associated with a *Cayley color graph*: the vertices correspond to the elements of the semigroup; next, imagine the generators of the semigroup to be associated with distinct colors. If vertices v_1 and v_2 correspond to semigroup elements s_1 and s_2 respectively, then there is a directed edge (of the color of the generator e) from v_1 to v_2 if and only if $s_1 e = s_2$. It is also possible to construct a Cayley color graph by action from the left. It is clear that for semigroups the structure of this graph may change heavily, when changing the side of the action.

In this note we consider the graph obtained from the Cayley color graph by suppressing all edge directions and all edge colors, deleting loops and multiple edges, that is, the uncolored Cayley graph. It is clear that in passing from the Cayley color graph to the corresponding uncolored graph algebraical information is lost but the genus is not changed. We call this graph *Cayley graph* and denote it by $Cay(S, C)$ for the semigroup S with the set of generators $C \subseteq S$.

The reader is referred to [1], [2], [3], [7], [10] and [11] for the terminology and notations which are not given in this paper.

We need the following results.

Result 1.1. (Euler, Poincaré 1758) *A finite graph with n vertices, m edges, which is 2-cell embedded on an orientable surface M of genus g with f faces fulfills the Euler-Poincaré formula: $n - m + f = 2 - 2g$.*

Result 1.2. (Maschke 1896) *The finite group G is planar if and only if $G = G_1 \times G_2$, where $G_1 = \mathbb{Z}_1$ or $\mathbb{Z}_2$ and $G_2 = \mathbb{Z}_n$, D_n, S_4, A_4 or A_5.*

Remark 1.3. It is clear that planarity depends on the set of generators C chosen for the Cayley graph. For example $Cay(\mathbb{Z}_6, \{1\}) = C_6$ and also $Cay(\mathbb{Z}_6, \{2, 3\})$ which is the box product $C_3 \Box K_2$ is planar, but $Cay(\mathbb{Z}_6, \{1, 2, 3\}) = K_6$ is not. For the planar groups D_n, S_4, A_4 or A_5 we get various Archemedian solids as Cayley graph representations, with two or three generators [9].

Result 1.4. (Kuratowski 1930) *A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

Result 1.5. (Chartrand, Harary 1967) *A finite graph is outer planar if and only if it does not contain a subgraph that is a subdivision of K_4 or $K_{2,3}$.*

2 The *Cay*-functor and right groups

For most of the considerations we can use the following two results which we take from [5]. However, as far as we know, there do not exist general formulas which relate the genus of a cross product or a lexicographic product of two graphs to the genera of the factors, compare for example [1], [2] or [10]. Some of the difficulties with respect to the lexicographic product can be seen in Example 3.8. We denote by $\times$ the *cross product* for graphs and also the direct product for semigroups and sets. By $X[Y]$ we denote the *lexicographic product* of the graph X with the graph Y.

Proposition 2.1. *For semigroups S and T with subsets C and D, respectively, we have $Cay(S \times T, C \times D) = Cay(S, C) \times Cay(T, D)$.*

Note that if in the above formula the semigroup T is R_r its graph $Cay(R_r, R_r)$ has to be considered as $K_r^{(r)}$, i. e. the complete graph with r loops.

Proposition 2.2. *Let S be a monoid with identity 1_S, T a semigroup, C and D subsets of S and T respectively. Then*

$$Cay(S \times T, (C \times T) \cup (\{1_S\} \times D)) = Cay(S, C)[Cay(T, D)]$$

if and only if $tT = T$ for any $t \in T$, that is if and only if T is a right group.

Remark 2.3. A formal description of the relation between graphs and subgraphs which are subdivisions with the help of the *Cay*-functor on semigroups with generators seems to be difficult. In $Cay(\mathbb{Z}_6, \{1\})$ we find a subdivision of K_3 corresponding to $Cay(\{0, 2, 4\}, \{2\})$, as a subgraph. But subdivision is not a categorical concept. And there is no inclusion between $\{0, 2, 4\} \times \{2\}$ and $\mathbb{Z}_6 \times \{1\}$.

3 The embeddings

Now we determine the minimal genus among the Cayley graphs $Cay(G \times R_r, C \times R_r)$ taken over all minimum generating set C of the group G. We do not claim that an embedding of this graph gives the (minimal) genus of the right group considered. Generally $G \times R_r$ may have a generating system $C' \neq C \times R_r$ which yields a Cayley graph with fewer edges and consequently tends to have a smaller genus. A straight-forward calculation yields the following lemma. Note that the first equality can also be obtained by applying Proposition 2.2 in the form $Cay(G \times R_r, (C \times R_r) \cup (\{1_G\} \times \emptyset)) = Cay(G, C)[Cay(R_r, \emptyset)]$.

Lemma 3.1. *Denote by $Cay(G, C)[\overline{K}_r]$ the lexicographic product of $Cay(G, C)$ with r isolated vertices. We have $Cay(G \times R_r, C \times R_r) = Cay(G, C)[\overline{K}_r]$.*

Note that this product can be seen as replacing every vertex of $Cay(G, C)$ by r independent vertices and every edge by a $K_{r,r}$. In particular $K_{k,k}[\overline{K}_r] = K_{kr,kr}$.

Proposition 3.2. *If $Cay(G, C)$ is not planar then $Cay(G \times R_r, C \times R_r)$ with $r \geq 2$ cannot be embedded on the torus.*

Proof. Already $K_{3,3}[\overline{K}_2] \cong K_{6,6}$ has genus 4. Moreover, the graph $K_5[\overline{K}_2]$ has 10 vertices and 40 edges. An embedding on the torus would have 30 faces by the formula of Euler-Poincaré. Even if all faces were triangles in this graph, this would require 45 edges. So the graphs are not toroidal. □

Proposition 3.3. *If $r \geq 5$ then $Cay(G \times R_r, C \times R_r)$ cannot be embedded on the torus.*

Proof. The resulting graph contains $K_{5,5}$ which has genus 3, compare [10]. □

Proposition 3.4. *If $Cay(G, C)$ contains a $K_{2,2}$ subdivision and $r \geq 3$ then $Cay(G \times R_r, C \times R_r)$ cannot be embedded on the torus.*

Proof. The resulting graph contains $K_{6,6}$ which has genus 4, compare [10]. □

Hence, for the rest of the paper we will check all planar groups G and $1 \leq r \leq 4$ for $Cay(G \times R_r, C \times R_r)$ having genus 1.

Lemma 3.5. *If the vertex degree of a planar $Cay(G, C)$ is at least 3 then $Cay(G \times R_2, C \times R_2)$ cannot be embedded on the torus.*

Proof. Since $Cay(G, C)$ is at least 3-regular $Cay(G \times R_2, C \times R_2)$ is at least 6-regular.

Assume that $Cay(G \times R_2, C \times R_2)$ is embedded on the torus, then the formula of Euler-Poincaré yields that all faces are triangular. This implies that every edge of $Cay(G \times R_2, C \times R_2)$ lies in at least two triangles, hence every edge of $Cay(G, C)$ lies in at least one triangle.

Let $c_1, c_2, c_3 \in C$ the generators corresponding to a triangle a_1, a_2, a_3. Then $c_1^{\pm 1} c_2^{\pm 1} c_3^{\pm 1} = 1_G$ for some signing, where 1_G is the identity in G. If any two of the c_i are distinct then one of the two is redundant, hence C was not inclusion minimal. Thus every $c \in C$ must be of order 3. Since G is not cyclic we obtain that $Cay(G, C)$ is at least 4-regular. The formula of Euler-Poincaré yields that the at least 8-regular $Cay(G \times R_2, C \times R_2)$ cannot be embedded on the torus. □

Theorem 3.6. *Let $G \times R_r$ be a finite right group with $r \geq 2$. The minimal genus of $Cay(G \times R_r, C \times R_r)$ among all generating sets $C \subseteq G$ of G is 1 iff $G \times R_r$ is isomorphic to one of the following rightgroups:*

- $\mathbb{Z}_n \times R_r$ *with* $(n, r) \in \{(2, 3), (2, 4), (3, 3), (i, 2)\}$ *for* $i \geq 4$
- $D_n \times R_2$ *for all* $n \geq 2$

Note that this list includes $\mathbb{Z}_2 \times D_n \times R_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_n \times R_2$ for odd $n \geq 3$.

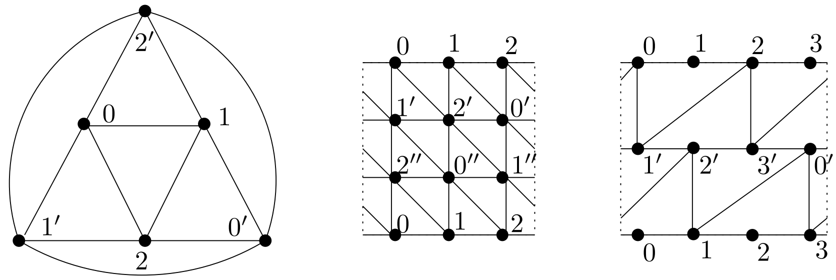

Figure 1: The planar $Cay(\mathbb{Z}_3 \times R_2, \{1\} \times R_2)$, the toroidal $Cay(\mathbb{Z}_3 \times R_3, \{1\} \times R_3)$ and $Cay(\mathbb{Z}_4 \times R_2, \{1\} \times R_2) \cong K_{4,4}$

Proof. By Lemma 3.5 the group G has to be either generated by one element or by two elements of order 2 in order to be embeddable on the torus. This necessary condition is equivalent to (G, C) being $(\mathbb{Z}_n, \{1\})$ or $(D_n, \{g_1, g_2\})$, where $g_1^2 = g_2^2 = (g_1 g_2)^n = e$.

First we consider the cyclic case. For $n = 2$ we have $Cay(\mathbb{Z}_2 \times R_r, C \times R_r) = K_{r,r}$ which exactly for $r \in \{3, 4\}$ has genus 1.

Take $n = 3$. If $r = 2$ we obtain the planar graph $Cay(\mathbb{Z}_3 \times R_2, \{1\} \times R_2)$ shown in Figure 1. If $r = 3$ the resulting graph contains $K_{3,3}$, so it cannot be planar. Figure 1 shows an embedding as a triangular grid on the torus. If $r = 4$ we have the complete tripartite graph $K_{4,4,4}$. Delete the entire set of 16 edges between two of the partitioning sets. The remaining (non-planar) graph has 12 vertices, 32 edges and, assuming a toroidal embedding, 20 faces. A simple count shows that this cannot be realized without traingular faces. So for $r \geq 4$ the graph $Cay(\mathbb{Z}_3 \times R_r, C \times R_r)$ is not toroidal.

Take $n \geq 4$. Now the graph $Cay(\mathbb{Z}_n, \{1\})$ contains a $C_4 = K_{2,2}$ subdivision. If $r \geq 3$ then $Cay(\mathbb{Z}_n \times R_r, \{1\} \times R_r)$ is not toroidal by Proposition 3.4. If $r = 2$ an embedding of $Cay(\mathbb{Z}_4 \times R_2, \{1\} \times R_2)$ as a square grid in the torus is shown in Figure 1. This is instructive for the cases $n \geq 5$. Moreover we see that the vertices $\{0, 0', 2\}$ and $\{1, 1', 3\}$ induce a $K_{3,3}$ subgraph of $Cay(\mathbb{Z}_4 \times R_2, \{1\} \times R_2)$. Generally for $n \geq 4$ we have that $Cay(\mathbb{Z}_n \times R_2, \{1\} \times R_2)$ contains a $K_{3,3}$ subdivision, it hence is not planar.

Second, if G is a dihedral group and C consists of two generators g_1, g_2 of order 2 the graph $Cay(D_n, C)$ is isomorphic to $Cay(\mathbb{Z}_{2n}, \{1\})$. Thus $Cay(D_n \times R_2, \{g_1, g_2\} \times R_r)$ has genus 1 if and only if $r = 2$ by the cyclic case. Any different generating system C for D_n would have a generator of degree greater than 2 and hence would yield $Cay(D_n \times R_2, C \times R_2)$ with genus greater than 1 by Lemma 3.5. □

Remark 3.7. *For the case $r = 1$ we have $G \times R_r \cong G$. Hence the above above theorem for $r = 1$ is the characterization of toroidal groups due to Proulx [8].*

In the above proofs we make strong use of Lemma 3.5, which tells us that

3-regular planar Cayley graphs will not be embeddable on the torus after taking the cartesian product with R_2. In fact, this operation can increase the genus from 0 to 3 already in the following small example.

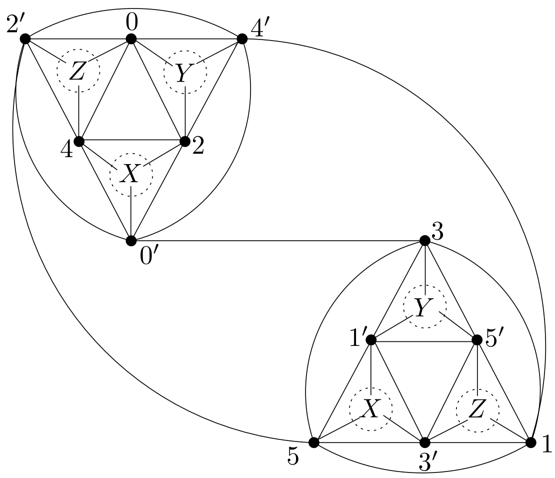

Figure 2: $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ in the triple torus with handles X, Y, Z.

Example 3.8. *The genus of* $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ *is* 3. *Note that* $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2) \cong (C_3 \square K_2)[\overline{K}_2]$.

Proof. To see this we observe that $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ consist of two disjoint copies $C_3 \square K_2$ and $(C_3 \square K_2)'$ of $Cay(\mathbb{Z}_6, \{2,3\})$ with vertex sets $\{0,1,2,3,4,5\}$ and $\{0',1',2',3',4',5'\}$, respectively. Every vertex v of $C_3 \square K_2$ is adjacent to every neighbor of its copy v' in $(C_3 \square K_2)'$. Figure 2 shows an embedding of $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ into the orientable surface of genus 3 – *the triple torus*. This graph is 6-regular with 12 vertices, so it has 36 edges.

By Lemma 3.5 $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ cannot be embedded on the torus.

So assume that $Cay(\mathbb{Z}_6 \times R_2, \{2,3\} \times R_2)$ is 2-cell-embedded on the double torus. Delete the 4 edges between $1, 1'$ and $5, 5'$ and the 4 edges between $0, 0'$ and $4, 4'$. The resulting graph H has 28 edges. It consists of two graphs A and B, which are copies of $K_{4,4}$, where A has the bipartition $(\{0,0',5,5'\}, \{2,2',3,3'\})$ and B has $(\{0,0',1,1'\}, \{3,3',4,4'\})$. They are glued at the four vertices with the same numbers and the corresponding 4 edges are identified. Although H is no longer bipartite it still is triangle-free. Hence by our assumption it is 2-cell-embedded on the double torus. By the formula of Euler-Poincaré this gives 14 faces and consequently all of them are quadrangular. So the edges between $1, 1'$ and $5, 5'$ and between $0, 0'$ and $4, 4'$, which we have to put back in, have to be diagonals of these quadrangular faces. But then $\{2', 4, 2, 0\}$ and $\{2', 4, 2, 0'\}$ are the only 4-cycles in H which contain the vertices $4, 0$ and $4, 0'$, respectively, they form faces of H. Since they have the common edges $\{2', 4\}$ and $\{2, 4\}$ we obtain a $K_{2,3}$ with bipartition $(\{2, 2'\}, \{0, 0', 4\})$. It is folklore that $K_{2,3}$ is not outer planar. Thus the region consisting of the glued 4-cycles $\{2', 4, 2, 0\}$ and $\{2', 4, 2, 0'\}$ must contain one of the vertices $0, 0'$ or 4 in its interior. Hence this vertex has only degree 2 – a contradiction. □

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